Numerical Computing
Tutorial 2

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1. The Lax-Wendroff approximation to the convection equation

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$$

is

$$U_j^{n+1} = U_j^n - \frac{a\Delta t}{2\Delta x}(U_{j+1}^n - U_{j-1}^n) + \frac{a^2\Delta t^2}{2\Delta x^2}(U_{j+1}^n - 2U_j^n + U_{j-1}^n).$$

(a) By performing a Taylor series expansion about $(j\Delta x, n\Delta t)$, show that the truncation error is $O(\Delta x^2, \Delta t^2)$.

(b) Using Fourier analysis, determine the timestep stability limit. (Hint: use double-angle formulae to reduce all expressions to functions of $\theta/2$.)

Answer

(a) Writing $u_k^m \equiv u(k\Delta x, m\Delta t)$, the truncation error is defined by:

$$T_j^n = \frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{a}{2\Delta x}(u_{j+1}^n - u_{j-1}^n) - \frac{a^2\Delta t}{2\Delta x^2}(u_{j+1}^n - 2u_j^n + u_{j-1}^n).$$

By Taylor series expansion (where u and its derivatives are assumed to be evaluated at $(j\Delta x, n\Delta t)$), we have:

$$\begin{aligned}
u_{j\pm1}^n &= u \pm \Delta x u_x + \frac{\Delta x^2}{2}u_{xx} \pm \frac{\Delta x^3}{6}u_{xxx} + \frac{\Delta x^4}{24}u_{xxxx} + O(\Delta x^5) \\
u_{j+1}^n - u_{j-1}^n &= 2\Delta x u_x + \frac{\Delta x^3}{3}u_{xxx} + O(\Delta x^5) \\
u_{j+1}^n + u_{j-1}^n &= 2u + \Delta x^2 u_{xx} + \frac{\Delta x^4}{12}u_{xxxx} + O(\Delta x^6) \\
u_j^{n+1} &= u + \Delta t u_t + \frac{\Delta t^2}{2}u_{tt} + \frac{\Delta t^3}{6}u_{ttt} + \frac{\Delta t^4}{24}u_{tttt} + O(\Delta t^5)
\end{aligned}$$

Thus

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = u_t + \frac{\Delta t}{2}u_{tt} + \frac{\Delta t^2}{6}u_{ttt} + \frac{\Delta t^3}{24}u_{tttt} + O(\Delta t^4)$$

and

$$\frac{a}{2\Delta x}(u_{j+1}^n - u_{j-1}^n) = au_x + \frac{a\Delta x^2}{6}u_{xxx} + O(\Delta x^4)$$

and

$$\begin{aligned}
\frac{a^2\Delta t}{2\Delta x^2}(u_{j+1}^n - 2u_j^n + u_{j-1}^n) &= \frac{a^2\Delta t}{2\Delta x^2}\left(\Delta x^2 u_{xx} + \frac{\Delta x^4}{12}u_{xxxx} + O(\Delta x^6)\right) \\
&= \frac{a^2\Delta t}{2}u_{xx} + \frac{a^2\Delta t\Delta x^2}{24}u_{xxxx} + O(\Delta t\Delta x^4).
\end{aligned}$$

Now, since $u_t + au_x = 0$, we note (as per the lecture notes) that:

$$\begin{aligned}
u_{tt} &= (u_t)_t \\
&= (-au_x)_t \\
&= (-au_t)_x \\
&= a^2 u_{xx}
\end{aligned}$$

Combining all the above terms to get the truncation error, therefore, we calculate that

$$\begin{aligned}
T_j^n &= u_t + \frac{\Delta t}{2}u_{tt} + \frac{\Delta t^2}{6}u_{ttt} + \frac{\Delta t^3}{24}u_{tttt} + O(\Delta t^4) \\
&+ au_x + \frac{a\Delta x^2}{6}u_{xxx} + O(\Delta x^4) \\
&- \left\{\frac{a^2\Delta t}{2}u_{xx} + \frac{a^2\Delta t\Delta x^2}{24}u_{xxxx} + O(\Delta t\Delta x^4)\right\} \\
&= \underbrace{u_t + au_x}_{=0} + \frac{\Delta t}{2}\underbrace{(u_{tt} - a^2u_{xx})}_{=0} \\
&+ \frac{\Delta t^2}{6}u_{ttt} + \frac{\Delta t^3}{24}u_{tttt} + O(\Delta t^4) + \frac{a\Delta x^2}{6}u_{xxx} + O(\Delta x^4) \\
&- \left\{\frac{a^2\Delta t\Delta x^2}{24}u_{xxxx} + O(\Delta t\Delta x^4)\right\} \\
&= \frac{\Delta t^2}{6}u_{ttt} + \frac{a\Delta x^2}{6}u_{xxx} + O(\Delta t^3) + O(\Delta x^4) + O(\Delta t\Delta x^2) \\
&= O(\Delta x^2, \Delta t^2),
\end{aligned}$$

as required.

(b) Writing $U_j^n = z^n e^{ij\theta}$, as usual, we have

$$\begin{aligned}
z &= 1 - \frac{a\Delta t}{2\Delta x}(e^{i\theta} - e^{-i\theta}) + \frac{a^2\Delta t^2}{2\Delta x^2}(e^{i\theta} - 2 + e^{-i\theta}) \\
&= 1 - \frac{a\Delta t}{\Delta x}i\sin\theta + \frac{a^2\Delta t^2}{\Delta x^2}(\cos\theta - 1) \\
&= 1 - \frac{a\Delta t}{\Delta x}\left(2i\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right) + \frac{a^2\Delta t^2}{\Delta x^2}\left(-2\sin^2\frac{\theta}{2}\right).
\end{aligned}$$

Writing $\tau = \theta/2$ and $k = a\Delta t/\Delta x > 0$ for increased brevity, we have

$$z = 1 - 2ki\sin\tau\cos\tau - 2k^2\sin^2\tau.$$

We want to determine when $|z| \le 1$, i.e.

$$\begin{aligned}
1 &\ge |z|^2 \\
&= (1 - 2k^2\sin^2\tau)^2 + (-2k\sin\tau\cos\tau)^2 \\
&= 1 - 4k^2\sin^2\tau + 4k^4\sin^4\tau + 4k^2\sin^2\tau\cos^2\tau \\
&= 1 + 4k^2\sin^2\tau\left(-1 + k^2\sin^2\tau + \cos^2\tau\right).
\end{aligned}$$

Note that when $k = 1$, i.e. $a\Delta t = \Delta x$, this is just equal to 1. In general, we need this to be less than or equal to 1, however, and since $4k^2\sin^2\tau \ge 0$, clearly our condition reduces to

$$\begin{aligned}
0 &\ge -1 + k^2\sin^2\tau + \cos^2\tau \\
&= -1 + k^2\sin^2\tau + (1 - \sin^2\tau) \\
&= (k^2 - 1)\sin^2\tau.
\end{aligned}$$

This in turn just reduces to $k^2 - 1 \le 0$, since $\sin^2\tau \ge 0$. So our condition is $k^2 \le 1$, or

$$\Delta t^2 \le \frac{\Delta x^2}{a^2}.$$

Since Δt, Δx and a are all ≥ 0, this simplifies to

$$\Delta t \le \frac{\Delta x}{a}.$$

2. A two-step predictor-corrector approximation to the convection equation

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$$

is

$$U_j^{(p)} = U_j^n - \frac{a\Delta t}{2\Delta x}(U_{j+1}^n - U_{j-1}^n)$$
$$U_j^{n+1} = U_j^n - \frac{a\Delta t}{2\Delta x}(U_{j+1}^{(p)} - U_{j-1}^{(p)}).$$

Using Fourier analysis, determine the timestep stability limit.

(Hint: let $U_j^n = z^n e^{ij\theta}$ and $U_j^{(p)} = Cz^n e^{ij\theta}$, and find the complex constant C and then z.)

Answer

Using the hint, we calculate that

$$\begin{aligned} C &= 1 - \frac{a\Delta t}{2\Delta x}(e^{i\theta} - e^{-i\theta}) \\ &= 1 - \frac{a\Delta t}{\Delta x} i \sin\theta \end{aligned}$$

and

$$\begin{aligned} z &= 1 - \frac{a\Delta t}{2\Delta x}(Ce^{i\theta} - Ce^{-i\theta}) \\ &= 1 - \frac{a\Delta t}{\Delta x}\left(1 - \frac{a\Delta t}{\Delta x} i\sin\theta\right) i\sin\theta \\ &= 1 - \frac{a\Delta t}{\Delta x} i\sin\theta + \frac{a^2\Delta t^2}{\Delta x^2} i^2 \sin^2\theta \\ &= 1 - \frac{a\Delta t}{\Delta x} i\sin\theta - \frac{a^2\Delta t^2}{\Delta x^2}\sin^2\theta. \end{aligned}$$

Now, writing $k = a\Delta t/\Delta x > 0$ for brevity, we have

$$z = 1 - ki\sin\theta - k^2\sin^2\theta.$$

We want to find when $|z| \leq 1$, i.e. when

$$\begin{aligned} 1 &\geq |z|^2 \\ &= \left(1 - k^2\sin^2\theta\right)^2 + (-k\sin\theta)^2 \\ &= 1 - 2k^2\sin^2\theta + k^4\sin^4\theta + k^2\sin^2\theta \\ &= 1 - k^2\sin^2\theta + k^4\sin^4\theta \end{aligned}$$

This is true when

$$k^4 \sin^4 \theta - k^2 \sin^2 \theta = k^2 \sin^2 \theta . (k^2 \sin^2 \theta - 1) \leq 0,$$

i.e. when

$$k^2 \sin^2 \theta \leq 1,$$

since (the other) $k^2 \sin^2 \theta \geq 0$. We want the tightest possible constraint on k, so we consider when $\sin^2 \theta = 1$. Then our condition is $k^2 \leq 1$, or

$$a^2 \Delta t^2 \leq \Delta x^2.$$

Noting that a, Δt and Δx are all positive, this again reduces to

$$\Delta t \leq \frac{\Delta x}{a}.$$

3. Let $u(x, y)$ be the solution of the steady 2D diffusion equation

$$\nabla^2 u \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

in the region between the circles $r = 1$ and $r = 2$ (where $r^2 = x^2 + y^2$) with the value of $u(x, y)$ specified on the two circular boundaries.

We wish to construct an approximate solution $U_{j,k}$ on a polar grid with

$$x_{j,k} = (1 + j\Delta r)\cos(k\Delta\theta),\ y_{j,k} = (1 + j\Delta r)\sin(k\Delta\theta),$$

with $\Delta r = 1/N$, $\Delta\theta = 2\pi/N$.

(a) First, think of this problem from a finite volume point of view. Construct a suitable quadrilateral control volume around each interior grid point. Write down the integral form of the steady diffusion equation for this control volume, and then form a discrete approximation by approximating each of the flux terms crossing the boundary of the control volume.

(b) Next, think of the problem from a finite difference point of view. In polar coordinates,

$$\nabla^2 u \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

Using this, write down a suitable finite difference approximation of the diffusion equation expressed in polar coordinates.

(c) Show that the finite volume scheme from part (a) is also a consistent finite difference approximation to the p.d.e. in polar coordinates. Is the converse true? (i.e. can your solution to part (b) be viewed as a finite volume method?)

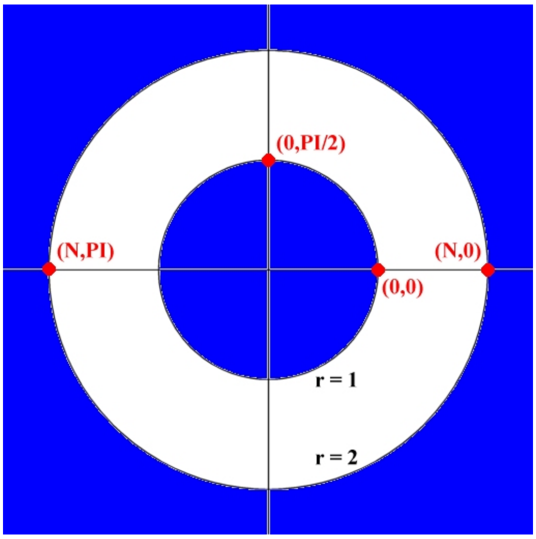
(d) How would you handle the periodic boundary condition at $\theta = 0, 2\pi$?

Answer

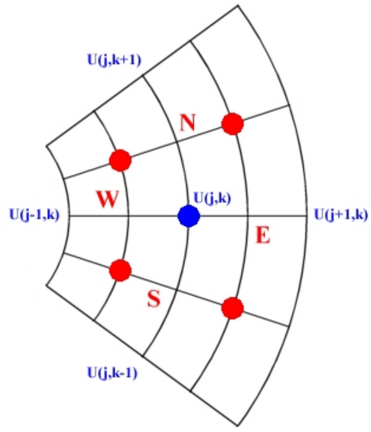
(a) See the diagram on the next page for a better indication of what's going on. Consider the point $U_{j,k}$ on the polar grid. We construct the following quadrilateral control volume around it: $(j + \frac{1}{2}, k + \frac{1}{2})$, $(j + \frac{1}{2}, k - \frac{1}{2})$, $(j - \frac{1}{2}, k - \frac{1}{2})$, $(j - \frac{1}{2}, k + \frac{1}{2})$.

Denote the face from $(j + \frac{1}{2}, k + \frac{1}{2})$ to $(j + \frac{1}{2}, k - \frac{1}{2})$ as $\mathcal{E}$, and the others (in the order given above) as $\mathcal{S}$, $\mathcal{W}$ and $\mathcal{N}$ respectively. The $\mathcal{N}$ and $\mathcal{S}$ faces are of length Δr and the others are of length $r\Delta\theta$ (for two different values of r). We calculate as follows (where $r_j \equiv 1 + j\Delta r$):

$$\begin{aligned}
& \oint \frac{\partial u}{\partial n}\, ds \\
= \ & \int_{\mathcal{E}} \frac{\partial u}{\partial n}\, ds + \int_{\mathcal{S}} \frac{\partial u}{\partial n}\, ds + \int_{\mathcal{W}} \frac{\partial u}{\partial n}\, ds + \int_{\mathcal{N}} \frac{\partial u}{\partial n}\, ds \\
\approx \ & \int_{\mathcal{E}} \frac{\partial u}{\partial r}\, ds + \int_{\mathcal{S}} \frac{\partial u}{\partial \theta}\, ds + \int_{\mathcal{W}} \frac{\partial u}{\partial r}\, ds + \int_{\mathcal{N}} \frac{\partial u}{\partial \theta}\, ds \\
\approx \ & r_{j+1/2}\Delta\theta \frac{U_{j+1,k} - U_{j,k}}{\Delta r} + \Delta r \frac{U_{j,k-1} - U_{j,k}}{r_j \Delta\theta} \\
+ \ & r_{j-1/2}\Delta\theta \frac{U_{j-1,k} - U_{j,k}}{\Delta r} + \Delta r \frac{U_{j,k+1} - U_{j,k}}{r_j \Delta\theta} \\
= \ & 0.
\end{aligned}$$



(i) The polar grid - we're solving in the unshaded (white) region



(ii) The quadrilateral volume (marked in red) around an interior grid point (in blue)

Figure 1: Visualising the problem domain and our finite volume method

(b) Expanding, we get

$$\begin{aligned}\nabla^2 u &\equiv \frac{1}{r}\left(r\frac{\partial^2 u}{\partial r^2}+\frac{\partial u}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}\\ &= \frac{\partial^2 u}{\partial r^2}+\frac{\partial u}{\partial r}\cdot\frac{1}{r}+\frac{\partial^2 u}{\partial\theta^2}\cdot\frac{1}{r^2}.\end{aligned}$$

Now, we know that at $U_{j,k}$, $r = 1 + j\Delta r$, so using central differences, our scheme then turns out to be

$$\frac{1}{\Delta r^2}\delta_r^2 U_{j,k}+\frac{1}{r}\cdot\frac{1}{2\Delta r}\cdot\delta_{2r}U_{j,k}+\frac{1}{\Delta\theta^2}\cdot\frac{1}{r^2}\cdot\delta_\theta^2 U_{j,k}=0.$$

Expanding this, we get

$$\frac{U_{j+1,k}-2U_{j,k}+U_{j-1,k}}{\Delta r^2}+\frac{1}{r}\cdot\frac{U_{j+1,k}-U_{j-1,k}}{2\Delta r}+\frac{1}{r^2}\frac{U_{j,k+1}-2U_{j,k}+U_{j,k-1}}{\Delta\theta^2}=0.$$

(c) i. To be consistent, the truncation error for our finite volume scheme must, by definition, tend to 0 as Δr, $\Delta\theta \to 0$. Accordingly, we start by determining the truncation error for our scheme, namely

$$\begin{aligned}\tau_{j,k} &= r_{j+1/2}\Delta\theta\frac{u_{j+1,k}-u_{j,k}}{\Delta r}+\Delta r\frac{u_{j,k-1}-u_{j,k}}{r_j\Delta\theta}\\ &+ r_{j-1/2}\Delta\theta\frac{u_{j-1,k}-u_{j,k}}{\Delta r}+\Delta r\frac{u_{j,k+1}-u_{j,k}}{r_j\Delta\theta},\end{aligned}$$

where $u_{j,k} \equiv u(x_{j,k}, y_{j,k})$. In what follows, u and its derivatives are assumed to be evaluated at $(x_{j,k}, y_{j,k})$ unless otherwise specified.

Now, Taylor expansion gives us that

$$\begin{aligned} u_{j\pm1,k} &= u \pm \Delta r\frac{\partial u}{\partial r} + \frac{\Delta r^2}{2}\frac{\partial^2 u}{\partial r^2} \pm \frac{\Delta r^3}{6}\frac{\partial^3 u}{\partial r^3} + \frac{\Delta r^4}{24}\frac{\partial^4 u}{\partial r^4} + O(\Delta r^5) \\ u_{j,k\pm1} &= u \pm \Delta\theta\frac{\partial u}{\partial \theta} + \frac{\Delta\theta^2}{2}\frac{\partial^2 u}{\partial \theta^2} \pm \frac{\Delta\theta^3}{6}\frac{\partial^3 u}{\partial \theta^3} + \frac{\Delta\theta^4}{24}\frac{\partial^4 u}{\partial \theta^4} + O(\Delta\theta^5), \end{aligned}$$

whence

$$\begin{aligned} \tau_{j,k} &= r_{j+1/2}\Delta\theta\left(\frac{\partial u}{\partial r} + \frac{\Delta r}{2}\frac{\partial^2 u}{\partial r^2} + \frac{\Delta r^2}{6}\frac{\partial^3 u}{\partial r^3} + \frac{\Delta r^3}{24}\frac{\partial^4 u}{\partial r^4} + O(\Delta r^4)\right) \\ &+ \frac{\Delta r}{r_j}\left(-\frac{\partial u}{\partial \theta} + \frac{\Delta\theta}{2}\frac{\partial^2 u}{\partial \theta^2} - \frac{\Delta\theta^2}{6}\frac{\partial^3 u}{\partial \theta^3} + \frac{\Delta\theta^3}{24}\frac{\partial^4 u}{\partial \theta^4} + O(\Delta\theta^4)\right) \\ &+ r_{j-1/2}\Delta\theta\left(-\frac{\partial u}{\partial r} + \frac{\Delta r}{2}\frac{\partial^2 u}{\partial r^2} - \frac{\Delta r^2}{6}\frac{\partial^3 u}{\partial r^3} + \frac{\Delta r^3}{24}\frac{\partial^4 u}{\partial r^4} + O(\Delta r^4)\right) \\ &+ \frac{\Delta r}{r_j}\left(\frac{\partial u}{\partial \theta} + \frac{\Delta\theta}{2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\Delta\theta^2}{6}\frac{\partial^3 u}{\partial \theta^3} + \frac{\Delta\theta^3}{24}\frac{\partial^4 u}{\partial \theta^4} + O(\Delta\theta^4)\right) \end{aligned}$$

Now, $r_{j\pm1/2}\Delta\theta \to 1 \times 0$ as $\Delta r,\ \Delta\theta \to 0$. The bracketed parts of the first and third terms tend to $\partial u/\partial r$ and $-\partial u/\partial r$ respectively as $\Delta r \to 0$, so the whole first and third terms tend to 0 overall as $\Delta r,\ \Delta\theta \to 0$. Furthermore, as $\Delta r \to 0$, $\Delta r/r_j \to 0/1 = 0$. The bracketed parts of the second and fourth terms tend to $-\partial u/\partial\theta$ and $\partial u/\partial\theta$ respectively as $\Delta\theta \to 0$, so the whole second and fourth terms tend to 0 overall as $\Delta r,\ \Delta\theta \to 0$. We thus observe that the whole truncation error tends to 0 as Δr and $\Delta\theta$ do, so the scheme is a consistent one.

ii. I'm not absolutely sure whether the solution to part (b) can be viewed as a finite volume method, but my gut feeling is no, because it contains approximations to second derivatives. The finite volume method uses flux terms crossing the various bits of the boundary, which will be first derivatives.

(d) The values at $\theta = 0$ depend on those at $\theta = \pm\Delta\theta$. Similarly, the values at $\theta = 2\pi$ depend on those at $\theta = 2\pi \pm \Delta\theta$. We note, however, that the values at $\theta = -\Delta\theta$ are the same as those at $\theta = 2\pi - \Delta\theta$, and that those at $\theta = 2\pi + \Delta\theta$ are the same as those at $\theta = \Delta\theta$. Thus we replace any terms in the scheme for the boundary cases with $k = -1$ with ones with $k = N-1$, and similarly we replace any terms with $k = N+1$ with ones with $k = 1$. For example, at $\theta = 0$, the first scheme would be

$$r_{j+1/2}\Delta\theta\frac{U_{j+1,0} - U_{j,0}}{\Delta r} + \Delta r\frac{U_{j,N-1} - U_{j,0}}{r_j\Delta\theta} + r_{j-1/2}\Delta\theta\frac{U_{j-1,0} - U_{j,0}}{\Delta r} + \Delta r\frac{U_{j,1} - U_{j,0}}{r_j\Delta\theta} = 0.$$

4. Let $u(x, y)$ be the solution of the steady 2D diffusion equation

$$\nabla^2 u \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

in the square $0 < x < 1$, $0 < y < 1$, with the value of $u(x, y)$ specified on the four sides of the square.

We wish to construct an approximate solution U_j on the grid shown above, using the piecewise linear finite element approximation presented in lectures. Note that U_5 is the only value to be determined; the others are all given by the boundary conditions.

(a) First, for the triangle 2-4-5, write down the linear approximation $U(x, y)$ using the nodal values U_2, U_4, U_5, and evaluate $\frac{\partial U}{\partial x}$, $\frac{\partial U}{\partial y}$, and hence

$$\int\int_{2-4-5} \left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial U}{\partial y}\right)^2 \, dx \, dy.$$

(b) Write down the corresponding contributions to

$$\int\int \left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial U}{\partial y}\right)^2 \, dx \, dy$$

from the other triangles, and derive the equation for U_5 by minimising this integral.

Answer

(a) We seek a linear approximation $U(x, y) = a + bx + cy$ such that

$$\begin{pmatrix} 1 & x_2 & y_2 \\ 1 & x_4 & y_4 \\ 1 & x_5 & y_5 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} U_2 \\ U_4 \\ U_5 \end{pmatrix}.$$

Now, assuming y increases downwards, $x_2 = 0.5$, $x_4 = 0$, $x_5 = 0.5$, $y_2 = 0$, $y_4 = 0.5$ and $y_5 = 0.5$. So our equation is

$$\begin{pmatrix} 1 & 0.5 & 0 \\ 1 & 0 & 0.5 \\ 1 & 0.5 & 0.5 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} U_2 \\ U_4 \\ U_5 \end{pmatrix}.$$

Solving this (with a graphics calculator or otherwise), we thus obtain

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 & 1 & -1 \\ 0 & -2 & 2 \\ -2 & 0 & 2 \end{pmatrix} \begin{pmatrix} U_2 \\ U_4 \\ U_5 \end{pmatrix} = \begin{pmatrix} U_2 + U_4 - U_5 \\ -2U_4 + 2U_5 \\ -2U_2 + 2U_5 \end{pmatrix},$$

and hence the linear approximation

$$U(x, y) = (U_2 + U_4 - U_5) + 2(U_5 - U_4)x + 2(U_5 - U_2)y.$$

Thus

$$\begin{aligned}\frac{\partial U}{\partial x} &= 2(U_5 - U_4)\\ \frac{\partial U}{\partial y} &= 2(U_5 - U_2),\end{aligned}$$

and hence

$$\begin{aligned}&\int\int_{2-4-5} \left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial U}{\partial y}\right)^2 \ dx\ dy\\ =\ & 4\left\{(U_5 - U_4)^2 + (U_5 - U_2)^2\right\} \int\int_{2-4-5} \ dx\ dy\\ =\ & 4\left\{(U_5 - U_4)^2 + (U_5 - U_2)^2\right\} \times \frac{1}{8}\\ =\ & \frac{1}{2}\left\{(U_5 - U_4)^2 + (U_5 - U_2)^2\right\}.\end{aligned}$$

(b) The contributions from the various triangles are as follows:

Triangle	$2\times$ Contribution
$1-2-4$	$(U_1 - U_2)^2 + (U_1 - U_4)^2$
$2-4-5$	$(U_5 - U_4)^2 + (U_5 - U_2)^2$
$2-3-5$	$(U_2 - U_3)^2 + (U_2 - U_5)^2$
$3-5-6$	$(U_6 - U_5)^2 + (U_6 - U_3)^2$
$4-5-7$	$(U_4 - U_5)^2 + (U_4 - U_7)^2$
$5-7-8$	$(U_8 - U_7)^2 + (U_8 - U_5)^2$
$5-6-8$	$(U_5 - U_6)^2 + (U_5 - U_8)^2$
$6-8-9$	$(U_9 - U_8)^2 + (U_9 - U_6)^2$

Adding up all the contributions, we thus get:

$$\begin{aligned}&\int\int \left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial U}{\partial y}\right)^2 \ dx\ dy\\ =\ & \frac{1}{2}(U_1 - U_2)^2 + \frac{1}{2}(U_1 - U_4)^2 + \frac{1}{2}(U_5 - U_4)^2 + \frac{1}{2}(U_5 - U_2)^2\\ +\ & \frac{1}{2}(U_2 - U_3)^2 + \frac{1}{2}(U_2 - U_5)^2 + \frac{1}{2}(U_6 - U_5)^2 + \frac{1}{2}(U_6 - U_3)^2\\ +\ & \frac{1}{2}(U_4 - U_5)^2 + \frac{1}{2}(U_4 - U_7)^2 + \frac{1}{2}(U_8 - U_7)^2 + \frac{1}{2}(U_8 - U_5)^2\\ +\ & \frac{1}{2}(U_5 - U_6)^2 + \frac{1}{2}(U_5 - U_8)^2 + \frac{1}{2}(U_9 - U_8)^2 + \frac{1}{2}(U_9 - U_6)^2.\end{aligned}$$

To minimise this, we observe that only terms which mention U_5 can be minimised here; the rest are fixed because of the boundary conditions. So minimising the above is equivalent to minimising:

$$\begin{aligned} & \frac{1}{2}(U_5 - U_4)^2 + \frac{1}{2}(U_5 - U_2)^2 + \frac{1}{2}(U_2 - U_5)^2 + \frac{1}{2}(U_6 - U_5)^2 \\ + & \frac{1}{2}(U_4 - U_5)^2 + \frac{1}{2}(U_8 - U_5)^2 + \frac{1}{2}(U_5 - U_6)^2 + \frac{1}{2}(U_5 - U_8)^2 \\ = & (U_5 - U_4)^2 + (U_5 - U_2)^2 + (U_5 - U_6)^2 + (U_5 - U_8)^2 \end{aligned}$$

Now, since the only 'variable' in this is U_5, we differentiate with respect to that and set the derivative to zero (note that I've elided the factors of 2 which arise from the differentiation):

$$(U_5 - U_4) + (U_5 - U_2) + (U_5 - U_6) + (U_5 - U_8) = 0$$

The equation for U_5 is thus simply

$$U_5 = \frac{U_2 + U_4 + U_6 + U_8}{4}.$$

This seems like a reassuring answer.