

# Numerical Analysis

## Problem Sheet 7

Stuart Golodetz

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1. Calculate the orthogonal polynomials  $\phi_0, \phi_1, \phi_2$  in

$$\langle f, g \rangle = \int_0^2 x f(x) g(x) dx$$

and hence solve again Question 7 from Problem Sheet 6.

**Answer**

(a) By the Lemma from the lectures, we have that:

$$\phi_0(x) = x^0 - 0 = 1$$

$$\begin{aligned} \lambda_0 \phi_0(x) &= \frac{\langle x, \phi_0 \rangle}{\langle \phi_0, \phi_0 \rangle} = \frac{\int_0^2 x \cdot x \cdot \phi_0(x) dx}{\int_0^2 x \cdot \phi_0(x) \cdot \phi_0(x) dx} \\ &= \frac{\int_0^2 x^2 dx}{\int_0^2 x dx} = \frac{[\frac{1}{3}x^3]_0^2}{[\frac{1}{2}x^2]_0^2} = \frac{4}{3} \end{aligned}$$

$$\phi_1(x) = x^1 - \lambda_0 \phi_0(x) = x - \frac{4}{3}$$

$$\lambda_0 \phi_0(x) = \frac{\langle x^2, \phi_0 \rangle}{\langle \phi_0, \phi_0 \rangle} = \frac{\int_0^2 x^3 dx}{2} = 2$$

$$\begin{aligned} \lambda_1 \phi_1(x) &= \frac{\langle x^2, \phi_1 \rangle}{\langle \phi_1, \phi_1 \rangle} \left(x - \frac{4}{3}\right) = \frac{\int_0^2 x^3 \left(x - \frac{4}{3}\right) dx}{\int_0^2 x \left(x - \frac{4}{3}\right)^2 dx} \left(x - \frac{4}{3}\right) \\ &= \frac{[\frac{1}{5}x^5 - \frac{4}{3}x^4]_0^2}{\int_0^2 x \left(x^2 - \frac{8}{3}x + \frac{16}{9}\right) dx} \left(x - \frac{4}{3}\right) = \frac{\frac{16}{15}}{[\frac{1}{4}x^4 - \frac{8}{9}x^3 + \frac{8}{9}x^2]_0^2} \left(x - \frac{4}{3}\right) \\ &= \frac{12}{5} \left(x - \frac{4}{3}\right) \end{aligned}$$

$$\phi_2(x) = x^2 - (\lambda_0 \phi_0(x) + \lambda_1 \phi_1(x)) = x^2 - \frac{12}{5} \left(x - \frac{4}{3}\right) - 2 = x^2 - \frac{12}{5}x + \frac{6}{5}$$

(b) Using the formula from the lecture notes, we get:

$$\begin{aligned} A &= \begin{pmatrix} \int_0^2 x \cdot \phi_0(x) \cdot \phi_0(x) dx & 0 & 0 \\ 0 & \int_0^2 x \cdot \phi_1(x) \cdot \phi_1(x) dx & 0 \\ 0 & 0 & \int_0^2 x \cdot \phi_2(x) \cdot \phi_2(x) dx \end{pmatrix} \\ &= \begin{pmatrix} \langle \phi_0, \phi_0 \rangle & 0 & 0 \\ 0 & \langle \phi_1, \phi_1 \rangle & 0 \\ 0 & 0 & \langle \phi_2, \phi_2 \rangle \end{pmatrix} \end{aligned}$$

Now as we've seen in the above calculations,  $\langle \phi_0, \phi_0 \rangle = 2$  and  $\langle \phi_1, \phi_1 \rangle = \frac{4}{9}$ , so it only remains to calculate  $\langle \phi_2, \phi_2 \rangle$ :

$$\int_0^2 x \left(x^2 - \frac{12}{5}x + \frac{6}{5}\right)^2 dx$$

Solving this numerically (because it looks long and boring to do analytically) and then converting to a fraction gives  $\frac{8}{75}$ . So:

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & \frac{4}{9} & 0 \\ 0 & 0 & \frac{8}{75} \end{pmatrix}$$

We now calculate  $f$ , again using the lecture notes:

$$f = \begin{pmatrix} \int_0^2 x \cdot x^3 \cdot \phi_0(x) dx \\ \int_0^2 x \cdot x^3 \cdot \phi_1(x) dx \\ \int_0^2 x \cdot x^3 \cdot \phi_2(x) dx \end{pmatrix} = \begin{pmatrix} \int_0^2 x^4 dx \\ \int_0^2 x^4 \cdot (x - \frac{4}{3}) dx \\ \int_0^2 x^4 \cdot (x^2 - \frac{12}{5}x + \frac{6}{5}) dx \end{pmatrix} = \begin{pmatrix} \frac{32}{5} \\ \frac{32}{15} \\ \frac{64}{175} \end{pmatrix}$$

Now if we solve  $A\alpha = f$  for  $\alpha$ , we get:

$$\alpha = \begin{pmatrix} 3.2 \\ 4.8 \\ 3.4286 \end{pmatrix}$$

In other words, the polynomial we're after is  $3.2\phi_0(x) + 4.8\phi_1(x) + 3.4286\phi_2(x) = 3.2 + 4.8(x - \frac{4}{3}) + 3.4286(x^2 - \frac{12}{5}x + \frac{6}{5}) = 0.9143 - 3.4286x + 3.4286x^2$ , which as expected (since the solution's unique) is the same as the answer we obtained by the old method.

2. If  $\{\phi_0, \phi_1, \dots, \phi_n, \dots\}$  are orthogonal polynomials in  $\langle \cdot, \cdot \rangle$  which are normalised to be monic (i.e. have leading coefficient equal to 1) show that  $\|\phi_k\| \leq \|q\|$  for all monic polynomials  $q \in \Pi_k$  which are of exact degree  $k$  where  $\|\cdot\|$  is the norm derived from the inner product.

### Answer

The orthogonal polynomials  $\{\phi_0, \dots, \phi_k\}$  form a basis for  $\Pi_k$ , so for some coefficients  $\alpha_i$ :

$$q(x) = \sum_{i=0}^k \alpha_i \phi_i(x) = \left( \sum_{i=0}^{k-1} \alpha_i \phi_i(x) \right) + \alpha_k \phi_k(x)$$

If we compute the inner product of  $q$  with itself, we get:

$$\langle q, q \rangle = \left\langle \sum_{i=0}^k \alpha_i \phi_i, \sum_{j=0}^k \alpha_j \phi_j \right\rangle = \sum_{i=0}^k \alpha_i \left\langle \phi_i, \sum_{j=0}^k \alpha_j \phi_j \right\rangle = \sum_{i=0}^k \alpha_i \sum_{j=0}^k \alpha_j \langle \phi_i, \phi_j \rangle = \sum_{i=0}^k \alpha_i^2 \langle \phi_i, \phi_i \rangle$$

We observe, since each  $\phi_i$  is monic and of exact degree  $i^1$ , that  $\alpha_k = 1$ , hence:

$$\langle q, q \rangle = \underbrace{\sum_{i=0}^{k-1} \alpha_i^2 \langle \phi_i, \phi_i \rangle}_{\geq 0} + \langle \phi_k, \phi_k \rangle$$

So  $\langle q, q \rangle \geq \langle \phi_k, \phi_k \rangle$  and hence  $\|q\| \geq \|\phi_k\|$ , as required.

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<sup>1</sup>Hopefully it's ok to assume this? I was a little unsure, since we're talking about a set of orthogonal polynomials: there doesn't appear to be anything to prevent the  $\phi_i$ s being rearranged so that  $\phi_k$  isn't of exact degree  $k$ .

4. If  $S$  is a linear spline interpolant of  $f$  on equally spaced points in  $[a, b]$  what is  $\int_a^b S(x) dx$ ?

**Answer**

It's just an application of the Composite Trapezium Rule we saw earlier in the course.

*Proof*

Suppose  $S$  interpolates  $f$  at equally-spaced points  $\{x_0, \dots, x_n\}$ . In other words,  $S(x_i) = f(x_i)$  for all  $i \in \{0, \dots, n\}$ . We know that if  $S_i$  is the part of the spline  $S$  approximating  $f$  between  $x_{i-1}$  and  $x_i$ , and  $h = x_i - x_{i-1} = \frac{b-a}{n}$ , then:

$$\begin{aligned} & \int_{x_{i-1}}^{x_i} S_i(x) dx \\ &= \frac{h}{2} (S(x_{i-1}) + S(x_i)) \\ &= \frac{h}{2} (f(x_{i-1}) + f(x_i)) \end{aligned}$$

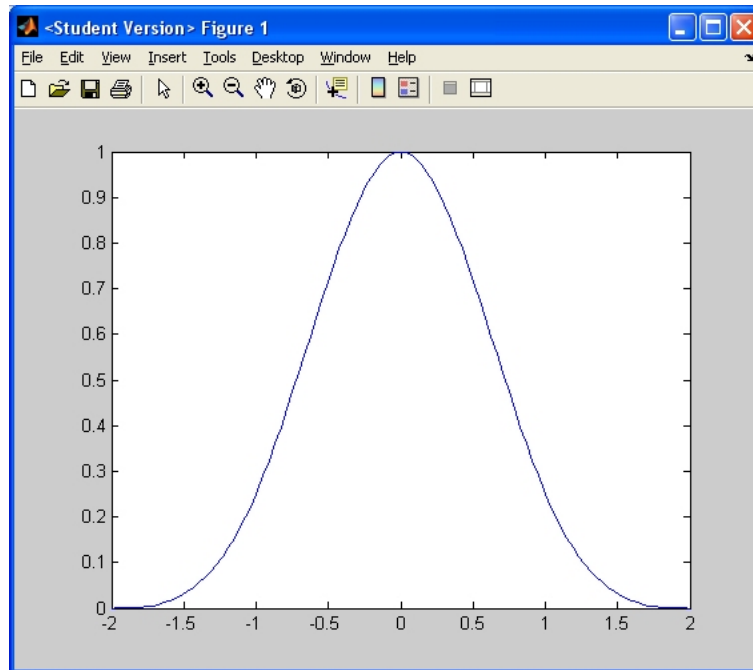
Now:

$$\begin{aligned} & \int_a^b S(x) dx \\ &= \sum_{i=1}^n \int_{x_{i-1}}^{x_i} S_i(x) dx \\ &= \frac{h}{2} \sum_{i=1}^n f(x_{i-1}) + f(x_i) \\ &= \frac{h}{2} [f(x_0) + 2f(x_1) + \dots + 2f(x_{n-1}) + f(x_n)] \end{aligned}$$

6. We use MATLAB as follows:

```
EDU>> x = [-2,-1,0,1,2]
x =
    -2    -1     0     1     2
EDU>> y = [0,0,1/4,1,1/4,0,0]
y =
     0     0  0.2500  1.0000  0.2500     0     0
EDU>> pp = spline(x,y)
pp =
    form: 'pp'
  breaks: [-2 -1 0 1 2]
   coefs: [4x4 double]
  pieces: 4
   order: 4
    dim: 1
EDU>> xs = linspace(-2,2);
EDU>> ys = ppval(pp,xs);
EDU>> plot(xs,ys)
```

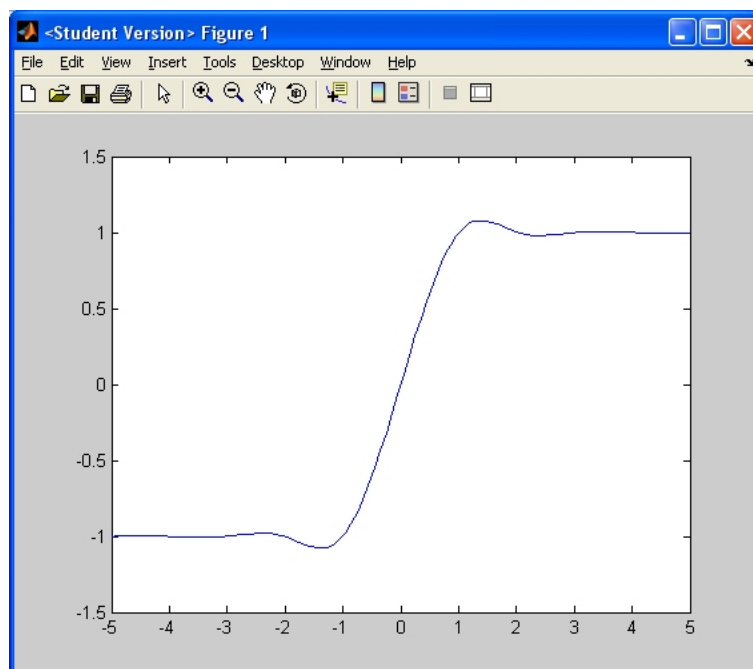
This gives us the following diagram:



7. The spline version is generated as follows:

```
EDU>> x = linspace(-5,5,11);  
EDU>> y(1) = 0; for i=1:11, y(i+1) = tanh(10*x(i)); end; y(13) = 0;  
EDU>> pp = spline(x,y);  
EDU>> xs = linspace(-5,5);  
EDU>> ys = ppval(pp,xs);  
EDU>> plot(xs,ys)
```

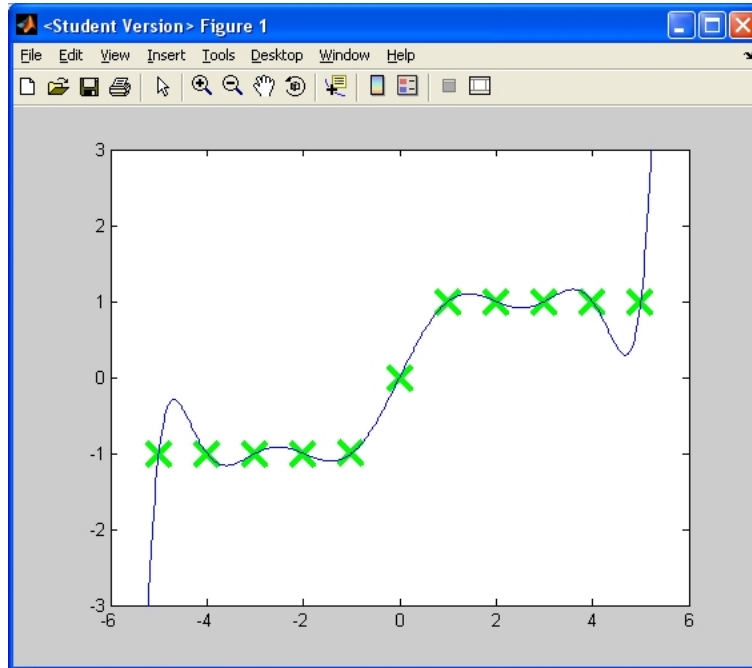
This produces the following image:



Contrast this to the Lagrange version, which is generated as follows:

```
EDU>> x = linspace(-5,5,11);
EDU>> for i=1:11, y(i) = tanh(10*x(i)); end;
EDU>> lagrange(x,y);
EDU>> axis([-6,6,-3,3])
```

This produces the following image:



Notice that the Lagrange version heads off to infinity outside the interpolation range, whereas the spline more sensibly stays near 1 on the positive side and  $-1$  on the negative side.

8. We want to show that:

$$\int_{x_0}^{x_n} S''(x)^2 dx \leq \int_{x_0}^{x_n} T''(x)^2 dx$$

The way to do this is to show that

$$\int_{x_0}^{x_n} (T''(x) - S''(x)) S''(x) dx = 0$$

since then we have

$$\begin{aligned} & \int_{x_0}^{x_n} T''(x)^2 - \int_{x_0}^{x_n} S''(x)^2 \\ &= \int_{x_0}^{x_n} (T''(x) - S''(x))^2 dx + 2 \int_{x_0}^{x_n} (T''(x) - S''(x)) S''(x) dx \\ &= \int_{x_0}^{x_n} (T''(x) - S''(x))^2 dx \\ &\geq 0 \end{aligned}$$

and the result follows directly. So:

$$\begin{aligned}
& \int_{x_0}^{x_n} (T''(x) - S''(x)) S''(x) dx \\
= & \sum_{k=1}^n \int_{x_{k-1}}^{x_k} \underbrace{(T''(x) - S''(x))}_{u'} \underbrace{S''(x)}_v dx \\
= & \sum_{k=1}^n \left( \left[ \underbrace{(T'(x) - S'(x))}_u \underbrace{S''(x)}_v \right]_{x_{k-1}}^{x_k} - \underbrace{\int_{x_{k-1}}^{x_k} (T'(x) - S'(x)) \underbrace{S'''(x)}_{v'} dx}_J \right)
\end{aligned}$$

Now integrating  $J$  by parts we get:

$$\begin{aligned}
& \int_{x_{k-1}}^{x_k} (T'(x) - S'(x)) S'''(x) dx \\
= & [(T(x) - S(x)) S'''(x)]_{x_{k-1}}^{x_k} - \int_{x_{k-1}}^{x_k} (T'(x) - S'(x)) \underbrace{S^{(4)}(x)}_{=0} dx \\
= & [(T(x) - S(x)) S'''(x)]_{x_{k-1}}^{x_k}
\end{aligned}$$

TODO: Sorry, I've run out of time to finish this one off!

10. (a) We observe that  $X \in V \Rightarrow X \in \Pi_3$  and  $X \in C^2[x_0, x_n]$  and  $X''(x_0) = X''(x_n) = 0$ .

We have to check the various vector space axioms:

- i. Vector addition is closed: If  $S \in V$  and  $T \in V$  then  $S + T \in V$ .

The sum of two cubic polynomials is clearly another cubic, so  $S + T \in \Pi_3$ . Furthermore, if  $S''(x_0) = S''(x_n) = T''(x_0) = T''(x_n)$  then:

$$(S + T)''(x_0) = S''(x_0) + T''(x_0) = 0 = S''(x_n) + T''(x_n) = (S + T)''(x_n)$$

It remains to show that  $S + T \in C^2[x_0, x_n]$ . Well, by the definition of  $C^k$  in the lecture notes, we have that  $f \in C^k[a, b]$  if  $f, f', \dots, f^{(k)}$  exist and are continuous on  $[a, b]$ . So  $S, S', S'', T, T', T''$  exist and are continuous on  $[x_0, x_n]$ . Thus  $S + T, (S + T)' = S' + T', (S + T)'' = S'' + T''$  exist and are continuous on  $[x_0, x_n]$ .

- ii. Scalar multiplication is closed: If  $\alpha \in \mathfrak{R}$  and  $S \in V$  then  $\alpha S \in V$ .

A scalar multiple of a cubic is clearly another cubic. Furthermore,  $(\alpha S)''(x_0) = \alpha S''(x_0) = 0$ . And  $\alpha S, (\alpha S)' = \alpha S', (\alpha S)'' = \alpha S''$  exist and are continuous on  $[x_0, x_n]$ .

- iii. Vector addition is associative: Trivially true due to the properties of polynomials.

- iv. Vector addition is commutative: Likewise.

- v. Vector addition has an identity element:  $0 \in \Pi_3$ , (abusing notation) we have that  $0''(x_0) = 0''(x_n) = 0$  and  $0 \in C^2[x_0, x_n]$  since (abusing notation)  $0, 0' = 0, 0'' = 0$  exist and are continuous on  $[x_0, x_n]$ .

- vi. Vector addition is invertible: For all  $S \in V$  there exists  $T \in V$  such that  $S + T = 0$ . Let  $T$  be the polynomial whose coefficients are the same as those of  $S$  but negated. Then clearly  $S + T = 0$ . Furthermore,  $T$  is in  $V$ : it's clearly in  $\Pi_3$ , its second derivative is clearly 0 when that of  $S$  is and it and its first and second derivatives are clearly defined and continuous on  $[x_0, x_n]$ .

- vii. Scalar multiplication is associative: Trivial.

- viii. Scalar multiplication has an identity: Trivial.

- ix. Distributivity holds for scalar multiplication over vector addition:  $\alpha(S + T) = \alpha S + \alpha T$ .

Trivially true due to the properties of polynomials.

- x. Distributivity holds for scalar multiplication over field addition:  $(\alpha + \beta)S = \alpha S + \beta S$ .  
Trivially true due to the properties of polynomials.

So  $V$  is definitely a vector space.

- (b) TODO: Had I had time to do Question 11, that would have constructively shown that the dimension of  $V$  is  $n + 1$ , since there's a basis  $B_0, \dots, B_n$  for it of size  $n + 1$ . Unfortunately I was a little short on time when doing this sheet! Is there a quicker way of showing it?
- (c) It's not a linear transformation from  $V$  to  $V$  because differentiating a cubic gives us a quadratic, so the result of differentiating isn't in  $V$ , which is the space of natural *cubic* splines.
- (d) The image of  $V$  under the operation of taking the second derivative is presumably the vector space of linear splines.