Numerical Analysis
Problem Sheet 7

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1. Calculate the orthogonal polynomials ϕ_0, ϕ_1, ϕ_2 in

$$\langle f, g\rangle = \int_0^2 x f(x) g(x)\; dx$$

and hence solve again Question 7 from Problem Sheet 6.

Answer

(a) By the Lemma from the lectures, we have that:

$$\begin{aligned}
\phi_0(x) &= x^0 - 0 = 1 \\
\lambda_0\phi_0(x) &= \frac{\langle x, \phi_0\rangle}{\langle \phi_0, \phi_0\rangle} = \frac{\int_0^2 x.x.\phi_0(x)\; dx}{\int_0^2 x.\phi_0(x).\phi_0(x)\; dx} \\
&= \frac{\int_0^2 x^2\; dx}{\int_0^2 x\; dx} = \frac{[\frac{1}{3}x^3]_0^2}{[\frac{1}{2}x^2]_0^2} = \frac{4}{3} \\
\phi_1(x) &= x^1 - \lambda_0\phi_0(x) = x - \tfrac{4}{3} \\
\lambda_0\phi_0(x) &= \frac{\langle x^2, \phi_0\rangle}{\langle \phi_0, \phi_0\rangle} = \frac{\int_0^2 x^3\; dx}{2} = 2 \\
\lambda_1\phi_1(x) &= \frac{\langle x^2, \phi_1\rangle}{\langle \phi_1, \phi_1\rangle}\left(x - \frac{4}{3}\right) = \frac{\int_0^2 x^3(x-\frac{4}{3})\; dx}{\int_0^2 x(x-\frac{4}{3})^2\; dx}\left(x - \frac{4}{3}\right) \\
&= \frac{\left[\frac{1}{5}x^5 - \frac{1}{3}x^4\right]_0^2}{\int_0^2 x(x^2 - \frac{8}{3}x + \frac{16}{9})\; dx}\left(x - \frac{4}{3}\right) = \frac{\frac{16}{15}}{\left[\frac{1}{4}x^4 - \frac{8}{9}x^3 + \frac{8}{9}x^2\right]_0^2}\left(x - \frac{4}{3}\right) \\
&= \frac{12}{5}\left(x - \frac{4}{3}\right) \\
\phi_2(x) &= x^2 - (\lambda_0\phi_0(x) + \lambda_1\phi_1(x)) = x^2 - \tfrac{12}{5}\left(x - \tfrac{4}{3}\right) - 2 = x^2 - \tfrac{12}{5}x + \tfrac{6}{5}
\end{aligned}$$

(b) Using the formula from the lecture notes, we get:

$$\begin{aligned}
A &= \begin{pmatrix} \int_0^2 x.\phi_0(x).\phi_0(x)\; dx & 0 & 0 \\ 0 & \int_0^2 x.\phi_1(x).\phi_1(x)\; dx & 0 \\ 0 & 0 & \int_0^2 x.\phi_2(x).\phi_2(x)\; dx \end{pmatrix} \\
&= \begin{pmatrix} \langle \phi_0, \phi_0\rangle & 0 & 0 \\ 0 & \langle \phi_1, \phi_1\rangle & 0 \\ 0 & 0 & \langle \phi_2, \phi_2\rangle \end{pmatrix}
\end{aligned}$$

Now as we've seen in the above calculations, $\langle \phi_0, \phi_0\rangle = 2$ and $\langle \phi_1, \phi_1\rangle = \frac{4}{9}$, so it only remains to calculate $\langle \phi_2, \phi_2\rangle$:

$$\int_0^2 x\left(x^2 - \frac{12}{5}x + \frac{6}{5}\right)^2\; dx$$

Solving this numerically (because it looks long and boring to do analytically) and then converting to a fraction gives $\frac{8}{75}$. So:

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & \frac{4}{9} & 0 \\ 0 & 0 & \frac{8}{75} \end{pmatrix}$$

We now calculate f, again using the lecture notes:

$$f = \begin{pmatrix} \int_0^2 x.x^3.\phi_0(x)\ dx \\ \int_0^2 x.x^3.\phi_1(x)\ dx \\ \int_0^2 x.x^3.\phi_2(x)\ dx \end{pmatrix} = \begin{pmatrix} \int_0^2 x^4\ dx \\ \int_0^2 x^4.(x - \frac{4}{3})\ dx \\ \int_0^2 x^4.(x^2 - \frac{12}{5}x + \frac{6}{5})\ dx \end{pmatrix} = \begin{pmatrix} \frac{32}{5} \\ \frac{32}{15} \\ \frac{64}{175} \end{pmatrix}$$

Now if we solve $A\alpha = f$ for α, we get:

$$\alpha = \begin{pmatrix} 3.2 \\ 4.8 \\ 3.4286 \end{pmatrix}$$

In other words, the polynomial we're after is $3.2\phi_0(x) + 4.8\phi_1(x) + 3.4286\phi_2(x) = 3.2 + 4.8\left(x - \frac{4}{3}\right) + 3.4286\left(x^2 - \frac{12}{5}x + \frac{6}{5}\right) = 0.9143 - 3.4286x + 3.4286x^2$, which as expected (since the solution's unique) is the same as the answer we obtained by the old method.

2. If $\{\phi_0, \phi_1, ..., \phi_n, ...\}$ are orthogonal polynomials in $\langle ., . \rangle$ which are normalised to be monic (i.e. have leading coefficient equal to 1) show that $||\phi_k|| \leq ||q||$ for all monic polynomials $q \in \Pi_k$ which are of exact degree k where $||.||$ is the norm derived from the inner product.

Answer

The orthogonal polynomials $\{\phi_0, ..., \phi_k\}$ form a basis for Π_k, so for some coefficients α_i:

$$q(x) = \sum_{i=0}^{k} \alpha_i \phi_i(x) = \left(\sum_{i=0}^{k-1} \alpha_i \phi_i(x) \right) + \alpha_k \phi_k(x)$$

If we compute the inner product of q with itself, we get:

$$\langle q, q \rangle = \left\langle \sum_{i=0}^{k} \alpha_i \phi_i, \sum_{j=0}^{k} \alpha_j \phi_j \right\rangle = \sum_{i=0}^{k} \alpha_i \left\langle \phi_i, \sum_{j=0}^{k} \alpha_j \phi_j \right\rangle = \sum_{i=0}^{k} \alpha_i \sum_{j=0}^{k} \alpha_j \langle \phi_i, \phi_j \rangle = \sum_{i=0}^{k} \alpha_i^2 \langle \phi_i, \phi_i \rangle$$

We observe, since each ϕ_i is monic and of exact degree i[1], that $\alpha_k = 1$, hence:

$$\langle q, q \rangle = \underbrace{\sum_{i=0}^{k-1} \alpha_i^2 \langle \phi_i, \phi_i \rangle}_{\geq 0} + \langle \phi_k, \phi_k \rangle$$

So $\langle q, q \rangle \geq \langle \phi_k, \phi_k \rangle$ and hence $||q|| \geq ||\phi_k||$, as required.

[1]Hopefully it's ok to assume this? I was a little unsure, since we're talking about a *set* of orthogonal polynomials: there doesn't appear to be anything to prevent the ϕ_is being rearranged so that ϕ_k isn't of exact degree k.

4. If S is a linear spline interpolant of f on equally spaced points in $[a,b]$ what is $\int_a^b S(x)\,dx$?

Answer

It's just an application of the Composite Trapezium Rule we saw earlier in the course.

Proof

Suppose S interpolates f at equally-spaced points $\{x_0, ..., x_n\}$. In other words, $S(x_i) = f(x_i)$ for all $i \in \{0, ..., n\}$. We know that if S_i is the part of the spline S approximating f between x_{i-1} and x_i, and $h = x_i - x_{i-1} = \frac{b-a}{n}$, then:

$$
\begin{aligned}
& \int_{x_{i-1}}^{x_i} S_i(x)\,dx \\
= & \frac{h}{2}\left(S(x_{i-1}) + S(x_i)\right) \\
= & \frac{h}{2}\left(f(x_{i-1}) + f(x_i)\right)
\end{aligned}
$$

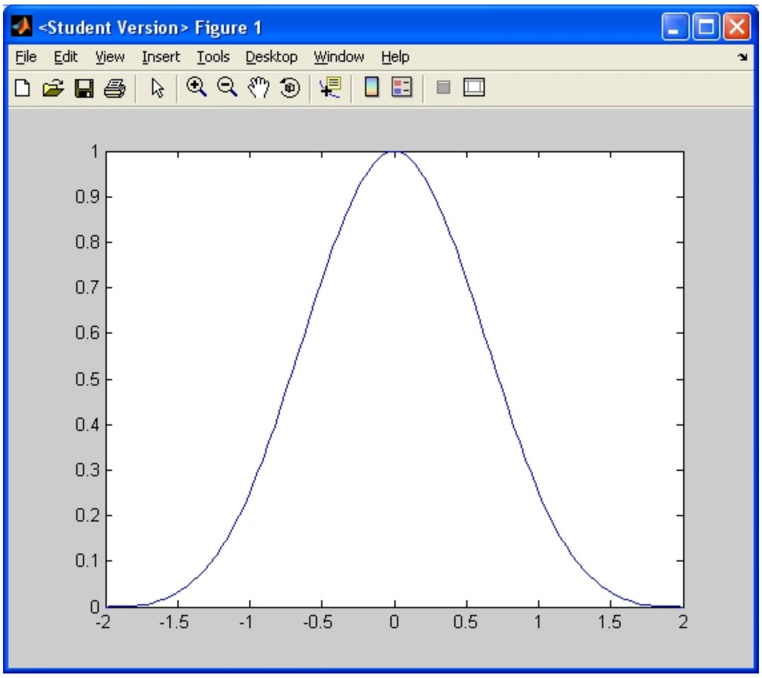
Now:

$$
\begin{aligned}
& \int_a^b S(x)\,dx \\
= & \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} S_i(x)\,dx \\
= & \frac{h}{2} \sum_{i=1}^{n} f(x_{i-1}) + f(x_i) \\
= & \frac{h}{2}\left[f(x_0) + 2f(x_1) + ... + 2f(x_{n-1}) + f(x_n)\right]
\end{aligned}
$$

6. We use MATLAB as follows:

```
EDU>> x = [-2,-1,0,1,2]

x =

    -2    -1     0     1     2

EDU>> y = [0,0,1/4,1,1/4,0,0]

y =

         0         0    0.2500    1.0000    0.2500         0         0

EDU>> pp = spline(x,y)

pp =

      form: 'pp'
    breaks: [-2 -1 0 1 2]
     coefs: [4x4 double]
    pieces: 4
     order: 4
       dim: 1

EDU>> xs = linspace(-2,2);
EDU>> ys = ppval(pp,xs);
EDU>> plot(xs,ys)
```

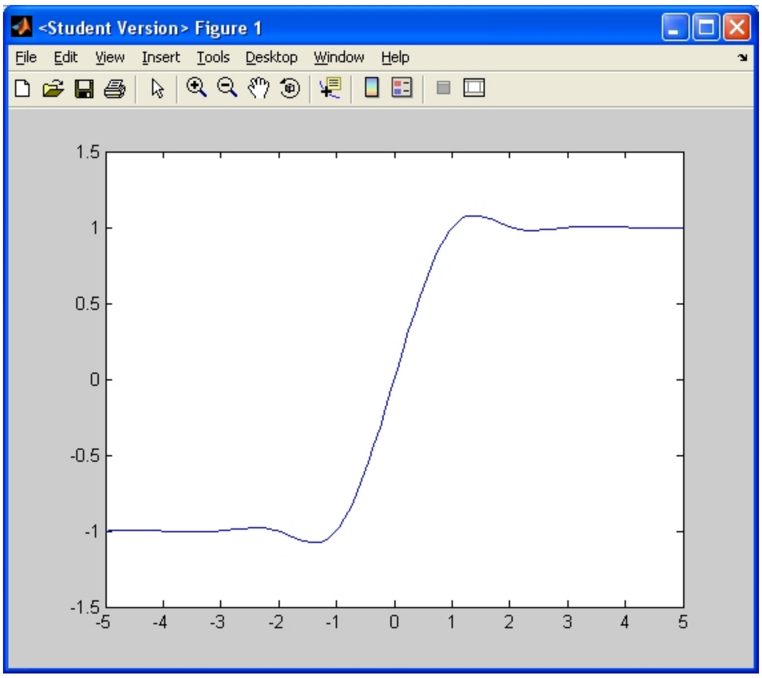
This gives us the following diagram:



7. The spline version is generated as follows:

```
EDU>> x = linspace(-5,5,11);
EDU>> y(1) = 0; for i=1:11, y(i+1) = tanh(10*x(i)); end; y(13) = 0;
EDU>> pp = spline(x,y);
EDU>> xs = linspace(-5,5);
EDU>> ys = ppval(pp,xs);
EDU>> plot(xs,ys)
```

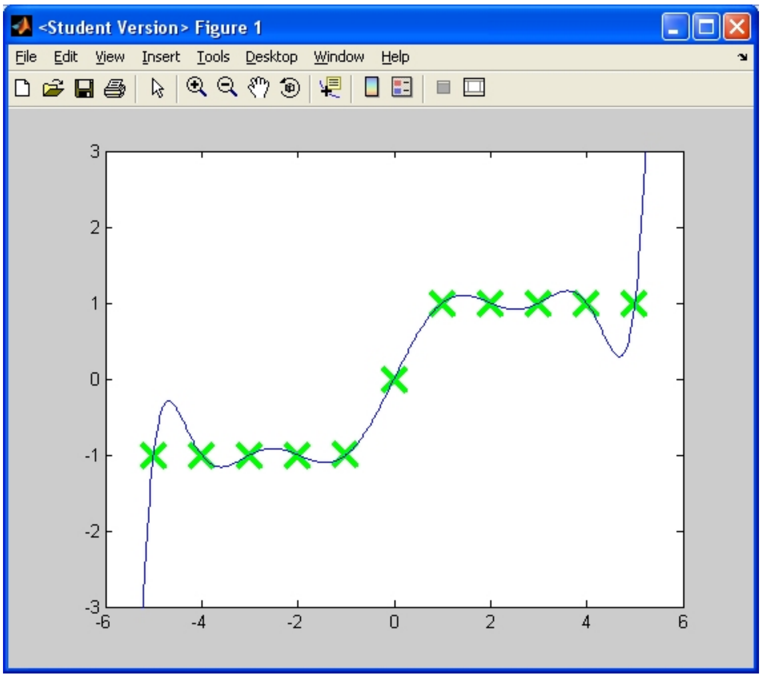
This produces the following image:



Contrast this to the Lagrange version, which is generated as follows:

```
EDU>> x = linspace(-5,5,11);
EDU>> for i=1:11, y(i) = tanh(10*x(i)); end;
EDU>> lagrange(x,y);
EDU>> axis([-6,6,-3,3])
```

This produces the following image:



Notice that the Lagrange version heads off to infinity outside the interpolation range, whereas the spline more sensibly stays near 1 on the positive side and -1 on the negative side.

8. We want to show that:

$$\int_{x_0}^{x_n} S''(x)^2 \, dx \leq \int_{x_0}^{x_n} T''(x)^2 \, dx$$

The way to do this is to show that

$$\int_{x_0}^{x_n} \left(T''(x) - S''(x)\right) S''(x) \, dx = 0$$

since then we have

$$\begin{aligned} & \int_{x_0}^{x_n} T''(x)^2 - \int_{x_0}^{x_n} S''(x)^2 \\ = & \int_{x_0}^{x_n} \left(T''(x) - S''(x)\right)^2 \, dx + 2\int_{x_0}^{x_n} \left(T''(x) - S''(x)\right) S''(x) \, dx \\ = & \int_{x_0}^{x_n} \left(T''(x) - S''(x)\right)^2 \, dx \\ \geq & \ 0 \end{aligned}$$

and the result follows directly. So:

$$\begin{aligned}
& \int_{x_0}^{x_n} \left(T''(x) - S''(x)\right) S''(x)\ dx \\
= & \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} \underbrace{\left(T''(x) - S''(x)\right)}_{u'} \underbrace{S''(x)}_{v}\ dx \\
= & \sum_{k=1}^{n} \left(\left[\underbrace{\left(T'(x) - S'(x)\right)}_{u} \underbrace{S''(x)}_{v} \right]_{x_{k-1}}^{x_k} - \underbrace{\int_{x_{k-1}}^{x_k} \underbrace{\left(T'(x) - S'(x)\right)}_{u} \underbrace{S'''(x)}_{v'}\ dx}_{J} \right)
\end{aligned}$$

Now integrating J by parts we get:

$$\begin{aligned}
& \int_{x_{k-1}}^{x_k} \left(T'(x) - S'(x)\right) S'''(x)\ dx \\
= & \left[\left(T(x) - S(x)\right) S'''(x)\right]_{x_{k-1}}^{x_k} - \int_{x_{k-1}}^{x_k} \left(T'(x) - S'(x)\right) \underbrace{S^{(4)}(x)}_{=0}\ dx \\
= & \left[\left(T(x) - S(x)\right) S'''(x)\right]_{x_{k-1}}^{x_k}
\end{aligned}$$

TODO: Sorry, I've run out of time to finish this one off!

10. (a) We observe that $X \in V \Rightarrow X \in \Pi_3$ and $X \in C^2[x_0, x_n]$ and $X''(x_0) = X''(x_n) = 0$.
We have to check the various vector space axioms:

i. Vector addition is closed: If $S \in V$ and $T \in V$ then $S + T \in V$.
The sum of two cubic polynomials is clearly another cubic, so $S + T \in \Pi_3$. Furthermore, if $S''(x_0) = S''(x_n) = T''(x_0) = T''(x_n)$ then:

$$(S+T)''(x_0) = S''(x_0) + T''(x_0) = 0 = S''(x_n) + T''(x_n) = (S+T)''(x_n)$$

It remains to show that $S+T \in C^2[x_0, x_n]$. Well, by the definition of C^k in the lecture notes, we have that $f \in C^k[a,b]$ if $f, f', ..., f^{(k)}$ exist and are continuous on $[a,b]$. So S, S', S'', T, T', T'' exist and are continuous on $[x_0, x_n]$. Thus $S + T, (S+T)' = S' + T', (S+T)'' = S'' + T''$ exist and are continuous on $[x_0, x_n]$.

ii. Scalar multiplication is closed: If $\alpha \in \Re$ and $S \in V$ then $\alpha S \in V$.
A scalar multiple of a cubic is clearly another cubic. Furthermore, $(\alpha S)''(x_0) = \alpha S''(x_0) = 0$. And $\alpha S, (\alpha S)' = \alpha S', (\alpha S)'' = \alpha S''$ exist and are continuous on $[x_0, x_n]$.

iii. Vector addition is associative: Trivially true due to the properties of polynomials.

iv. Vector addition is commutative: Likewise.

v. Vector addition has an identity element: $0 \in \Pi_3$, (abusing notation) we have that $0''(x_0) = 0''(x_n) = 0$ and $0 \in C^2[x_0, x_n]$ since (abusing notation) $0, 0' = 0, 0'' = 0$ exist and are continuous on $[x_0, x_n]$.

vi. Vector addition is invertible: For all $S \in V$ there exists $T \in V$ such that $S + T = 0$.
Let T be the polynomial whose coefficients are the same as those of S but negated. Then clearly $S + T = 0$. Furthermore, T is in V: it's clearly in Π_3, its second derivative is clearly 0 when that of S is and it and its first and second derivatives are clearly defined and continuous on $[x_0, x_n]$.

vii. Scalar multiplication is associative: Trivial.

viii. Scalar multiplication has an identity: Trivial.

ix. Distributivity holds for scalar multiplication over vector addition: $\alpha(S + T) = \alpha S + \alpha T$.
Trivially true due to the properties of polynomials.

x. Distributivity holds for scalar multiplication over field addition: $(\alpha+\beta)S = \alpha S+\beta S$. Trivially true due to the properties of polynomials.

So V is definitely a vector space.

(b) TODO: Had I had time to do Question 11, that would have constructively shown that the dimension of V is $n+1$, since there's a basis $B_0, ..., B_n$ for it of size $n+1$. Unfortunately I was a little short on time when doing this sheet! Is there a quicker way of showing it?

(c) It's not a linear transformation from V to V because differentiating a cubic gives us a quadratic, so the result of differentiating isn't in V, which is the space of natural *cubic* splines.

(d) The image of V under the operation of taking the second derivative is presumably the vector space of linear splines.