Numerical Analysis
Problem Sheet 6

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1. (a) We require that $\max_{x\in[a,b]} |f(x)| \geq 0$. This is trivial since we're taking the absolute value of something. We further require that when $f \equiv 0$, $\max_{x\in[a,b]} |f(x)| = 0$, which is trivially satisfied.

   (b) We require that $\max_{x\in[a,b]} |(\lambda f)(x)| = |\lambda| \max_{x\in[a,b]} |f(x)|$. Well $\max_{x\in[a,b]} |(\lambda f)(x)| = \max_{x\in[a,b]} |\lambda||f(x)| = |\lambda| \max_{x\in[a,b]} |f(x)|$.

   (c) Finally, we require that $\max_{x\in[a,b]} |(f+g)(x)| \leq \max_{x\in[a,b]} |f(x)| + \max_{x\in[a,b]} |g(x)|$. Consider that the LHS is equal to $|(f+g)(x_0)|$ for some $x_0 \in [a,b]$. Furthermore, the RHS is equal to $|f(x_1)| + |g(x_2)|$ for some $x_1, x_2 \in [a,b]$. Now since we have the maximums, it's clearly true that $|f(x_1)| + |g(x_2)| \geq |f(x_0)| + |g(x_0)|$. And by the triangle inequality property for the real numbers, $|f(x_0)| + |g(x_0)| \geq |(f+g)(x_0)|$. So the LHS is no greater than the RHS and we're done.

   So this defines a norm, as required.

2. See separate sheet.

3. (a) i. We want $\max_{1\leq i\leq n} |v_i| \geq 0$. This is obviously true because we're taking the maximum of a load of absolute values, which must all necessarily be non-negative. We also want that when $v = 0$, this is 0. Again, this is obviously true because then the absolute value of the largest element of v must be 0 as all its elements are 0.

      ii. We want:

$$\max_{1\leq i\leq n} |(\lambda v)_i| = |\lambda| \max_{1\leq i\leq n} |v_i|$$

      Well $\max_{1\leq i\leq n} |(\lambda v)_i| = |\lambda| v_j$ for some j. So $|\lambda| v_j \geq |\lambda| v_k$ for all $k \neq j$. So $v_j \geq v_k$ for all $k \neq j$. So $v_j = \max_{1\leq i\leq n} |v_i|$. So $|\lambda| \max_{1\leq i\leq n} |v_i| = |\lambda| v_j = \max_{1\leq i\leq n} |(\lambda v)_i|$.

      iii. Finally, we require that:

$$\max_{1\leq i\leq n} |v_i + w_i| \leq \max_{1\leq i\leq n} |v_i| + \max_{1\leq i\leq n} |w_i|$$

      We note that the LHS = $|v_j + w_j|$ for some j, that the RHS = $|v_k|$ + $|w_\ell|$ for some k, ℓ and that $|v_k| \geq |v_j|$ and $|w_\ell| \geq |w_j|$. Whence

$$|v_k| + |w_\ell| \geq |v_j| + |w_j| \geq |v_j + w_j|$$

      and we're done.

   So $||v||_\infty$ is a norm.

   (b) i. We want $\sum_{i=1}^n |v_i| \geq 0$. This is obviously true because we're summing a load of absolute values, which must all necessarily be non-negative. Furthermore, when $v = 0$, the sum evaluates to 0, as required.

      ii. We want $\sum_{i=1}^n |(\lambda v)_i| = |\lambda| \sum_{i=1}^n |v_i|$. This is obvious since:

$$\sum_{i=1}^n |(\lambda v)_i| = \sum_{i=1}^n |\lambda||v_i| = |\lambda| \sum_{i=1}^n |v_i|$$

      iii. Finally, we require that:

$$\sum_{i=1}^n |(v+w)_i| \leq \sum_{i=1}^n |v_i| + \sum_{i=1}^n |w_i|$$

      Well:

$$\sum_{i=1}^{n} |(v+w)_i| = \sum_{i=1}^{n} |v_i + w_i| \leq \sum_{i=1}^{n} |v_i| + |w_i| = \sum_{i=1}^{n} |v_i| + \sum_{i=1}^{n} |w_i|$$

So $||v||_1$ is a norm.

(c) For diagrams, see separate sheet.

4. (a) Yes, it's a convergent series.

   (b) Yes, it's a convergent series (though it doesn't actually do much converging!). MATLAB gives 1 as the answer for every value of r. This is as expected, since the analytical answer is:

$$\max_{1 \leq i \leq r} \left| \frac{1}{i} \right| = 1$$

   (c) No, it's not a convergent series. It's a harmonic series, which diverges to infinity (albeit slowly).

5. See separate sheet.

6. See separate sheet.

7. See separate sheet.

8. TODO

9. TODO: I couldn't figure out how to do this one, sorry.

10. (a)
```
EDU>> n = 3

n =

     3

EDU>> for i=0:n, for j=0:n, A(i+1,j+1) = 2^(i+j+2)/(i+j+2); end; end;
EDU>> A

A =

    2.0000    2.6667    4.0000    6.4000
    2.6667    4.0000    6.4000   10.6667
    4.0000    6.4000   10.6667   18.2857
    6.4000   10.6667   18.2857   32.0000

EDU>> f = [32/5; 64/6; 128/7; 256/8]

f =

    6.4000
   10.6667
   18.2857
   32.0000

EDU>> A\f

ans =

   -0.0000
   -0.0000
    0.0000
    1.0000

EDU>> clear
EDU>> n = 4
```

```
n =

     4

EDU>> for i=0:n, for j=0:n, A(i+1,j+1) = 2^(i+j+2)/(i+j+2); end; end;
EDU>> A

A =

    2.0000    2.6667    4.0000    6.4000   10.6667
    2.6667    4.0000    6.4000   10.6667   18.2857
    4.0000    6.4000   10.6667   18.2857   32.0000
    6.4000   10.6667   18.2857   32.0000   56.8889
   10.6667   18.2857   32.0000   56.8889  102.4000

EDU>> f = [32/5; 64/6; 128/7; 256/8; 512/9]

f =

    6.4000
   10.6667
   18.2857
   32.0000
   56.8889

EDU>> A\f

ans =

   -0.0000
    0.0000
   -0.0000
    1.0000
   -0.0000
```

Comments: In the Π_3 case, the polynomial is $0 + 0x + 0x^2 + 1x^3 = x^3$. In the Π_4 case, it's $0 + 0x + 0x^2 + 1x^3 + 0x^4 = x^3$. This isn't surprising: clearly the cubic polynomial which best approximates x^3 is itself.

(b) TODO