

Numerical Analysis

Problem Sheet 6

Stuart Golodetz

February 28, 2006

1. (a) We require that $\max_{x \in [a,b]} |f(x)| \geq 0$. This is trivial since we're taking the absolute value of something. We further require that when $f \equiv 0$, $\max_{x \in [a,b]} |f(x)| = 0$, which is trivially satisfied.
- (b) We require that $\max_{x \in [a,b]} |(\lambda f)(x)| = |\lambda| \max_{x \in [a,b]} |f(x)|$. Well $\max_{x \in [a,b]} |(\lambda f)(x)| = \max_{x \in [a,b]} |\lambda| |f(x)| = |\lambda| \max_{x \in [a,b]} |f(x)|$.
- (c) Finally, we require that $\max_{x \in [a,b]} |(f+g)(x)| \leq \max_{x \in [a,b]} |f(x)| + \max_{x \in [a,b]} |g(x)|$. Consider that the LHS is equal to $|(f+g)(x_0)|$ for some $x_0 \in [a,b]$. Furthermore, the RHS is equal to $|f(x_1)| + |g(x_2)|$ for some $x_1, x_2 \in [a,b]$. Now since we have the maximums, it's clearly true that $|f(x_1)| + |g(x_2)| \geq |f(x_0)| + |g(x_0)|$. And by the triangle inequality property for the real numbers, $|f(x_0)| + |g(x_0)| \geq |(f+g)(x_0)|$. So the LHS is no greater than the RHS and we're done.

So this defines a norm, as required.

2. See separate sheet.

3. (a) i. We want $\max_{1 \leq i \leq n} |v_i| \geq 0$. This is obviously true because we're taking the maximum of a load of absolute values, which must all necessarily be non-negative. We also want that when $v = 0$, this is 0. Again, this is obviously true because then the absolute value of the largest element of v must be 0 as all its elements are 0.
- ii. We want:

$$\max_{1 \leq i \leq n} |(\lambda v)_i| = |\lambda| \max_{1 \leq i \leq n} |v_i|$$

Well $\max_{1 \leq i \leq n} |(\lambda v)_i| = |\lambda| v_j$ for some j . So $|\lambda| v_j \geq |\lambda| v_k$ for all $k \neq j$. So $v_j \geq v_k$ for all $k \neq j$. So $v_j = \max_{1 \leq i \leq n} |v_i|$. So $|\lambda| \max_{1 \leq i \leq n} |v_i| = |\lambda| v_j = \max_{1 \leq i \leq n} |(\lambda v)_i|$.

iii. Finally, we require that:

$$\max_{1 \leq i \leq n} |v_i + w_i| \leq \max_{1 \leq i \leq n} |v_i| + \max_{1 \leq i \leq n} |w_i|$$

We note that the LHS = $|v_j + w_j|$ for some j , that the RHS = $|v_k| + |w_\ell|$ for some k, ℓ and that $|v_k| \geq |v_j|$ and $|w_\ell| \geq |w_j|$. Whence

$$|v_k| + |w_\ell| \geq |v_j| + |w_j| \geq |v_j + w_j|$$

and we're done.

So $\|v\|_\infty$ is a norm.

- (b) i. We want $\sum_{i=1}^n |v_i| \geq 0$. This is obviously true because we're summing a load of absolute values, which must all necessarily be non-negative. Furthermore, when $v = 0$, the sum evaluates to 0, as required.
- ii. We want $\sum_{i=1}^n |(\lambda v)_i| = |\lambda| \sum_{i=1}^n |v_i|$. This is obvious since:

$$\sum_{i=1}^n |(\lambda v)_i| = \sum_{i=1}^n |\lambda| |v_i| = |\lambda| \sum_{i=1}^n |v_i|$$

iii. Finally, we require that:

$$\sum_{i=1}^n |(v+w)_i| \leq \sum_{i=1}^n |v_i| + \sum_{i=1}^n |w_i|$$

Well:

$$\sum_{i=1}^n |(v+w)_i| = \sum_{i=1}^n |v_i + w_i| \leq \sum_{i=1}^n |v_i| + |w_i| = \sum_{i=1}^n |v_i| + \sum_{i=1}^n |w_i|$$

So $\|v\|_1$ is a norm.

(c) For diagrams, see separate sheet.

4. (a) Yes, it's a convergent series.

(b) Yes, it's a convergent series (though it doesn't actually do much converging!). MATLAB gives 1 as the answer for every value of r . This is as expected, since the analytical answer is:

$$\max_{1 \leq i \leq r} \left| \frac{1}{i} \right| = 1$$

(c) No, it's not a convergent series. It's a harmonic series, which diverges to infinity (albeit slowly).

5. See separate sheet.

6. See separate sheet.

7. See separate sheet.

8. TODO

9. TODO: I couldn't figure out how to do this one, sorry.

10. (a) `EDU>> n = 3`

`n =`

`3`

`EDU>> for i=0:n, for j=0:n, A(i+1,j+1) = 2^(i+j+2)/(i+j+2); end; end;`
`EDU>> A`

`A =`

```

2.0000    2.6667    4.0000    6.4000
2.6667    4.0000    6.4000   10.6667
4.0000    6.4000   10.6667   18.2857
6.4000   10.6667   18.2857   32.0000

```

`EDU>> f = [32/5; 64/6; 128/7; 256/8]`

`f =`

```

6.4000
10.6667
18.2857
32.0000

```

`EDU>> A\f`

`ans =`

```

-0.0000
-0.0000
0.0000
1.0000

```

`EDU>> clear`

`EDU>> n = 4`

```

n =
    4
EDU>> for i=0:n, for j=0:n, A(i+1,j+1) = 2^(i+j+2)/(i+j+2); end; end;
EDU>> A
A =
    2.0000    2.6667    4.0000    6.4000   10.6667
    2.6667    4.0000    6.4000   10.6667   18.2857
    4.0000    6.4000   10.6667   18.2857   32.0000
    6.4000   10.6667   18.2857   32.0000   56.8889
   10.6667   18.2857   32.0000   56.8889  102.4000
EDU>> f = [32/5; 64/6; 128/7; 256/8; 512/9]
f =
    6.4000
   10.6667
   18.2857
   32.0000
   56.8889
EDU>> A\f
ans =
   -0.0000
    0.0000
   -0.0000
    1.0000
   -0.0000

```

Comments: In the Π_3 case, the polynomial is $0 + 0x + 0x^2 + 1x^3 = x^3$. In the Π_4 case, it's $0 + 0x + 0x^2 + 1x^3 + 0x^4 = x^3$. This isn't surprising: clearly the cubic polynomial which best approximates x^3 is itself.

(b) TODO