Numerical Analysis
Problem Sheet 5

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1. (a) By Gershgorin's Theorem, the eigenvalues of the matrix lie in the discs:

$$\begin{aligned} D_1 &= \{z \in \mathbb{C} \mid \ |9 - z| \leq 1\} \\ D_2 &= \{z \in \mathbb{C} \mid \ |4 - z| \leq 1 + \epsilon\} \\ D_3 &= \{z \in \mathbb{C} \mid \ |1 - z| \leq \epsilon\} \end{aligned}$$

Since $|\epsilon| < 1$, the discs are mutually disjoint (consider that (i) the smallest thing in D_1 is 7 and the largest in D_2 is strictly less than 6 and (ii) the smallest thing in D_2 is strictly greater than 2 and the largest in D_3 is strictly less than 2). So by Gershgorin's Second Theorem there is exactly one eigenvalue in each disc. The best estimates I can come up with for the eigenvalues are hence 9, 4 and 1, the points in the middles of the three discs.

(b) Consider the diagonal similarity transformation:

$$\begin{aligned} & \underbrace{\begin{bmatrix} 1 & & \\ & 1 & \\ & & \epsilon \end{bmatrix}}_{P^{-1}} \begin{bmatrix} 9 & 1 & 0 \\ 1 & 4 & \epsilon \\ 0 & \epsilon & 1 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & & \\ & 1 & \\ & & \frac{1}{\epsilon} \end{bmatrix}}_{P} \\ = & \begin{bmatrix} 9 & 1 & 0 \\ 1 & 4 & \epsilon \\ 0 & \epsilon^2 & \epsilon \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & & \frac{1}{\epsilon} \end{bmatrix} \\ = & \underbrace{\begin{bmatrix} 9 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & \epsilon^2 & 1 \end{bmatrix}}_{A'} \end{aligned}$$

Now the eigenvalues of A' are the same as those of A, since the two matrices are similar. In particular, we now have that $|\lambda_3 - 1| \leq \epsilon^2$, since the Gershgorin disc for the third row of A' is $\{z \in \mathbb{C} \mid \ |1 - z| \leq \epsilon^2\}$.

2. An example would be

$$\begin{bmatrix} 1 & 10 \\ 0.1 & 2 \end{bmatrix}$$

the eigenvalues of which are roughly 0.3820 and 2.6180 (calculated using MATLAB). Both of these are clearly in the first disc (of size 10, centred on 1) and neither of them is in the second disc (of size 0.1, centred on 2).

3. (a) By Gershgorin's Theorem, the eigenvalues of the matrix lie in the discs:

$$\begin{aligned} D_1 &= \{z \in \mathbb{C} \mid \ |-2 - z| \leq 1 + 0 + 1\} = \{z \in \mathbb{C} \mid \ |-2 - z| \leq 2\} \\ D_2 &= \{z \in \mathbb{C} \mid \ |-1 - z| \leq 2 + 1 + 0\} = \{z \in \mathbb{C} \mid \ |-1 - z| \leq 3\} \\ D_3 &= \{z \in \mathbb{C} \mid \ |9 - z| \leq 1 + 0 + 1\} = \{z \in \mathbb{C} \mid \ |9 - z| \leq 2\} \\ D_4 &= \{z \in \mathbb{C} \mid \ |1 - z| \leq 2 + 0 + 1\} = \{z \in \mathbb{C} \mid \ |1 - z| \leq 3\} \end{aligned}$$

By Gershgorin's Second Theorem, since D_3 is disconnected from the other discs, it contains exactly 1 eigenvalue, i.e. λ_1 since that's the largest eigenvalue. The smallest λ_1 can be is hence $9 - 2 = 7$.

What about λ_2? Well, it can't be in D_3, so it must be in one of the other discs. The largest it can possibly be is 4, supposing it was in D_4 and as big as possible. So a lower bound for $\frac{|\lambda_1|}{|\lambda_2|}$ is $\frac{7}{4}$.

(b) After iteration k of the Power Method, we have $A^k x = \lambda_1^k \left[\alpha_1 v_1 + \sum_{i=2}^n \alpha_i \left(\frac{\lambda_i}{\lambda_1}\right)^k v_i\right]$. Now, given that $\alpha_1 = ... = \alpha_n = \alpha$ for some α, we can rewrite this as:

$$\lambda_1^k \alpha \left[v_1 + \sum_{i=2}^n \left(\frac{\lambda_i}{\lambda_1}\right)^k v_i\right]$$

We want to find the smallest k s.t. the term in the direction of v_1 is guaranteed to be at least 10^4 times bigger than the sum of the terms in the other eigenvalue directions, i.e. we want to solve:

$$\begin{aligned} & \lambda_1^k \alpha \geq 10^4 \lambda_1^k \alpha \textstyle\sum_{i=2}^n \left(\frac{\lambda_i}{\lambda_1}\right)^k \\ \Leftrightarrow \quad & 1 \geq 10^4 \textstyle\sum_{i=2}^n \left(\frac{\lambda_i}{\lambda_1}\right)^k \\ \Leftrightarrow \quad & 10^{-4} \geq \textstyle\sum_{i=2}^n \left(\frac{\lambda_i}{\lambda_1}\right)^k \end{aligned}$$

Now we know that $\frac{\lambda_2}{\lambda_1}$ is at most $\frac{4}{7}$ and that $\lambda_i \leq \lambda_2$ for all $i \geq 2$, so in particular we know that $\sum_{i=2}^n \left(\frac{\lambda_i}{\lambda_1}\right)^k \leq (n-1)\left(\frac{4}{7}\right)^k$. Now $n = 4$ here, so we know that:

$$\sum_{i=2}^n \left(\frac{\lambda_i}{\lambda_1}\right)^k \leq 3\left(\frac{4}{7}\right)^k$$

We want therefore that:

$$\begin{aligned} & 3\left(\tfrac{4}{7}\right)^k \leq 10^{-4} \\ \Leftrightarrow \quad & \log 3 + k \log \tfrac{4}{7} \leq \log 10^{-4} = -4 \\ \Leftrightarrow \quad & k \geq \left\lceil \frac{-4 - \log 3}{\log \frac{4}{7}} \right\rceil = 19 \end{aligned}$$

So we need at least 19 iterations of the Power Method to guarantee our result.

(c)
```
EDU>> A = [-2,1,0,1; 2,-1,-1,0; 1,0,9,-1; 2,0,-1,1]

A =

    -2     1     0     1
     2    -1    -1     0
     1     0     9    -1
     2     0    -1     1

EDU>> x=randn(4,1),x=x/sqrt(x'*x)

x =

   -0.4326
   -1.6656
    0.1253
    0.2877


x =

   -0.2473
   -0.9522
    0.0717
    0.1645

EDU>> y=A*x;x=y/sqrt(y'*y)

x =
```

```
   -0.4367
    0.5749
    0.3472
   -0.5985

EDU>> y=A*x;x=y/sqrt(y'*y)

x =

    0.2000
   -0.4225
    0.7735
   -0.4281

EDU>> y=A*x;x=y/sqrt(y'*y)

x =

   -0.1617
    0.0063
    0.9814
   -0.1037

EDU>> y=A*x;x=y/sqrt(y'*y)

x =

    0.0252
   -0.1459
    0.9765
   -0.1567

EDU>> y=A*x;x=y/sqrt(y'*y)

x =

   -0.0389
   -0.0860
    0.9884
   -0.1193

EDU>> y=A*x;x=y/sqrt(y'*y)

x =

   -0.0140
   -0.1076
    0.9855
   -0.1302

EDU>> y=A*x;x=y/sqrt(y'*y)

x =

   -0.0230
   -0.0995
    0.9868
   -0.1256

EDU>> y=A*x;x=y/sqrt(y'*y)

x =

   -0.0197
   -0.1025
    0.9864
   -0.1272

EDU>> y=A*x;x=y/sqrt(y'*y)

x =

   -0.0209
   -0.1014
    0.9865
```

```
   -0.1266

EDU>> y=A*x;x=y/sqrt(y'*y)

x =

   -0.0204
   -0.1018
    0.9865
   -0.1268

EDU>> y=A*x;x=y/sqrt(y'*y)

x =

   -0.0206
   -0.1016
    0.9865
   -0.1267

EDU>> y=A*x;x=y/sqrt(y'*y)

x =

   -0.0205
   -0.1017
    0.9865
   -0.1268

EDU>> y=A*x;x=y/sqrt(y'*y)

x =

   -0.0206
   -0.1017
    0.9865
   -0.1267

EDU>> y=A*x;x=y/sqrt(y'*y)

x =

   -0.0206
   -0.1017
    0.9865
   -0.1267

EDU>> y(3)/x(3)

ans =

    9.1076

EDU>> eig(A)

ans =

    9.1076
   -3.3765
    1.5950
   -0.3262
```

Looking at the eigenvalues of A, we can see that the calculated value (9.1076) is correct.

4. (a) If we apply the Power Method with the matrix $B = A^{-1}$, we compute a set of unit vectors in the directions x, Bx, B^2x, ..., B^kx, i.e. x, $A^{-1}x$, $(A^{-1})^2x$, ..., $(A^{-1})^kx$. Now, we observe that if A has a basis of eigenvectors with corresponding eigenvalues $\lambda_1, ..., \lambda_n$ then so does A^{-1}, specifically ones with corresponding eigenvalues $\frac{1}{\lambda_i}$ for $i \in \{1, ..., n\}$.

   (If this isn't obvious, consider the solutions of $A^{-1}w = \mu w$: by noting that if $Av_i = \lambda_i v_i$ then $A^{-1}(Av_i) = v_i = \mu_i(\lambda_i v_i)$, we can see that each $\mu_i = \frac{1}{\lambda_i}$, hence the observation above.)

Denote the basis of (normalised) eigenvectors for A^{-1} as w_i for $i \in \{1, ..., n\}$.
(Hence $\forall i \cdot ||w_i|| = 1$.) We can write (for some β_i):

$$x = \sum_{i=1}^{n} \beta_i w_i$$

where $A^{-1}w_i = \mu_i w_i$. We also assume that $\mu_1 > \mu_2 \geq ... \geq \mu_n$, i.e. there is some largest eigenvalue μ_1 of A^{-1} and hence some smallest eigenvalue $\frac{1}{\mu_1} = \lambda_1$ of A. So:

$$(A^{-1})^k x = (A^{-1})^k \sum_{i=1}^{n} \beta_i w_i = \sum_{i=1}^{n} \beta_i (A^{-1})^k w_i = \sum_{i=1}^{n} \beta_i \mu_i^k w_i$$

And rewriting similarly to the lecture notes:

$$(A^{-1})^k x = \mu_1^k \left[\beta_1 w_1 + \underbrace{\sum_{i=2}^{n} \beta_i \left(\frac{\mu_i}{\mu_1} \right)^k w_i}_{\to 0 \text{ } as \text{ } k \to \infty} \right]$$

So $(A^{-1})^k x$ tends to look like $\mu_1^k \beta_1 w_1$ as k gets large. So as per the lecture notes we have that:

$$\frac{||(A^{-1})^k x||}{||(A^{-1})^{k-1} x||} \approx \frac{|\mu_1^k \beta_1|}{|\mu_1^{k-1} \beta_1|} \to |\mu_1| = \frac{1}{\lambda_1} \text{ } as \text{ } k \to \infty$$

In other words, we converge to the reciprocal of λ_1, which as noted before is A's smallest eigenvalue (in terms of absolute value) in this instance.

(b) TODO

(c) Each time we're doing the computation $y = A^{-1}x$ in the loop. Note that we don't need to compute A^{-1} to do this if we rewrite as $Ay = x$, LU factorise A to give $LUy = x$ and then solve for y as normal using forward and back substitution.

5. (a) First consider the eigenvalues of $A - \mu I$. We observe that if $Av = \lambda v$ then $(A - \mu I)v = Av - \mu v = \lambda v - \mu v = (\lambda - \mu)v$, i.e. for each eigenvalue λ of A, $\lambda - \mu$ is an eigenvalue of $A - \mu I$. Now, if μ is an approximation to an eigenvalue λ_k of A s.t. $|\lambda_k - \mu| \leq |\lambda_i - \mu|$ for all $i \neq k$, then $\lambda_k - \mu$ is the smallest eigenvalue of $A - \mu I$. We know from question 4 that applying the Power Method to the inverse of a matrix B allows us to find the inverse of the smallest eigenvalue of B, so in this case applying the Power Method to $(A - \mu I)^{-1}$ allows us to find the inverse of the smallest eigenvalue of $A - \mu I$, in other words it allows us to find $\frac{1}{\lambda_k - \mu} = \gamma$. It's now easy to see that $\lambda_k = \mu + \frac{1}{\gamma}$, as the question pointed out. This, then, is the value of this method: instead of only allowing us to find the largest (or smallest) eigenvalue of A, this method allows us to find any eigenvalue of A provided we have an approximation to it that is closer to it than to any other eigenvalue of A (i.e. the solution converges to the eigenvalue nearest to our original approximation μ).

(b) TODO

6. (a) We calculate as follows:

$$
\begin{aligned}
& J(1,n)J(1,n-1)\cdots J(1,2)AJ(1,2)^T \\
= & \begin{bmatrix} \alpha & m_{12} & \cdots & m_{1n} \\ 0 & m_{22} & \cdots & m_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & m_{n2} & \cdots & m_{nn} \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & & & \\ \sin\theta & \cos\theta & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix} \\
= & \begin{bmatrix} \alpha\cos\theta + m_{12}\sin\theta & -\alpha\sin\theta + m_{12}\cos\theta & m_{13} & \cdots & m_{1n} \\ m_{22}\sin\theta & m_{22}\cos\theta & m_{23} & \cdots & m_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ m_{n2}\sin\theta & m_{n2}\cos\theta & m_{n3} & \cdots & m_{nn} \end{bmatrix}
\end{aligned}
$$

Well, the rightmost $n-2$ columns of this are certainly non-zero in general, since they're just the rightmost $n-2$ columns of the matrix $J(1,n)J(1,n-1)\cdots J(1,2)A$, whose entries are non-zero in general.

What about the other entries? Well, consider an entry $m_{k2}\sin\theta$: this is zero iff either $m_{k2} = 0$ or $\sin\theta = 0$. Neither of these is zero in general. The same goes for $m_{k2}\cos\theta$. The only remaining entries are $\alpha\cos\theta + m_{12}\sin\theta$ and $-\alpha\sin\theta + m_{12}\cos\theta$. Neither of these are zero in general.

(b) Consider for example:

$$
\begin{bmatrix} \alpha & \times & \cdots & \times \\ \beta & \times & \cdots & \times \\ 0 & \times & \cdots & \times \\ \vdots & \vdots & & \vdots \\ 0 & \times & \cdots & \times \end{bmatrix} \underbrace{\begin{bmatrix} 1 & & & & \\ & \cos\theta & -\sin\theta & & \\ & \sin\theta & \cos\theta & & \\ & & & 1 & \\ & & & & \ddots & \\ & & & & & 1 \end{bmatrix}}_{J(2,3)^T} = \begin{bmatrix} \alpha & \times & \cdots & \times \\ \beta & \times & \cdots & \times \\ 0 & \times & \cdots & \times \\ \vdots & \vdots & & \vdots \\ 0 & \times & \cdots & \times \end{bmatrix}
$$

Since the first column of any of the relevant $J(2,k)^T$ matrices is just $(1,0,...,0)^T$, the first column of the original matrix gets retained (we don't care at this stage what happens to the other columns). We can postmultiply by however many of them we like and the first column won't change. Thus the zeros in the first column don't get destroyed by the postmultiplications.

(Note: I got a bit confused by the comment about notation here, any chance of briefly explaining it in a sentence?!)

(c) The obvious way is just to observe that the matrix we get is equal to its transpose and hence symmetric (so if we get $(\alpha,\beta,0,...,0)^T$ as our first column, we get $(\alpha,\beta,0,...,0)$ as our first row). Using the fact that $(AB)^T = B^TA^T$, we just write:

$$
\begin{aligned}
& [J(2,n)\cdots J(2,3)AJ(2,3)^T\cdots J(2,n)^T]^T \\
= & (J(2,n)^T)^T\cdots (J(2,3)^T)^T A^T J(2,3)^T \cdots J(2,n)^T \\
= & J(2,n)\cdots J(2,3)AJ(2,3)^T\cdots J(2,n)^T \qquad \text{(note that } A^T = A\text{)}
\end{aligned}
$$

7. TODO

8. It suffices to show that $A_k = Q_kR_k + \mu_k I$ is similar to $A_{k+1} = R_kQ_k + \mu_k I$. Well:

$$A_{k+1} = R_kQ_k + \mu_k I = Q_k^T Q_k(R_kQ_k + \mu_k I) = Q_k^T(A_k - \mu_k I)Q_k + Q_k^T \mu_k I Q_k = Q_k^{-1} A_k Q_k$$

So A_{k+1} is similar to A_k and we're done. Note that, as required, this really doesn't depend on the various μ_k values.

9.
```
EDU>> A=randn(6,6), A=A+A'

A =

   -0.4326    1.1892   -0.5883   -0.0956   -0.6918   -0.3999
   -1.6656   -0.0376    2.1832   -0.8323    0.8580    0.6900
    0.1253    0.3273   -0.1364    0.2944    1.2540    0.8156
    0.2877    0.1746    0.1139   -1.3362   -1.5937    0.7119
   -1.1465   -0.1867    1.0668    0.7143   -1.4410    1.2902
    1.1909    0.7258    0.0593    1.6236    0.5711    0.6686


A =

   -0.8651   -0.4764   -0.4630    0.1920   -1.8382    0.7910
   -0.4764   -0.0753    2.5105   -0.6577    0.6713    1.4158
   -0.4630    2.5105   -0.2728    0.4083    2.3208    0.8749
    0.1920   -0.6577    0.4083   -2.6724   -0.8794    2.3355
   -1.8382    0.6713    2.3208   -0.8794   -2.8819    1.8614
    0.7910    1.4158    0.8749    2.3355    1.8614    1.3372

EDU>> B=hess(A)

B =

   -2.2139    0.7059         0         0         0         0
    0.7059   -1.3250    0.9839         0         0         0
         0    0.9839   -1.3365    2.4416         0         0
         0         0    2.4416    0.1286   -3.4362         0
         0         0         0   -3.4362   -2.0207   -3.5093
         0         0         0         0   -3.5093    1.3372

EDU>> eig(A)

ans =

   -6.0874
   -2.9672
   -2.2485
   -0.6883
    1.7775
    4.7835

EDU>> eig(B)

ans =

   -6.0874
   -2.9672
   -2.2485
   -0.6883
    1.7775
    4.7835

EDU>> [Q,R] = qr(B)

Q =

   -0.9527   -0.2215   -0.0281   -0.0948   -0.0653   -0.1708
    0.3038   -0.6946   -0.0881   -0.2973   -0.2049   -0.5358
         0    0.6845   -0.0985   -0.3324   -0.2291   -0.5990
         0         0    0.9908   -0.0621   -0.0428   -0.1120
         0         0         0    0.8878   -0.1644   -0.4298
         0         0         0         0   -0.9340    0.3572


R =
```

```
    2.3237   -1.0751    0.2989         0         0         0
         0    1.4374   -1.5982    1.6712         0         0
         0         0    2.4642   -0.1130   -3.4048         0
         0         0         0   -3.8704   -1.5805   -3.1156
         0         0         0         0    3.7571   -0.6722
         0         0         0         0         0    1.9860

EDU>> R*Q

ans =

   -2.5405    0.4367    0.0000    0.0000    0.0000    0.0000
    0.4367   -2.0923    1.6866   -0.0000         0    0.0000
         0    1.6866   -0.3546   -3.8350         0   -0.0000
         0         0   -3.8350   -1.1626    3.3356    0.0000
         0         0         0    3.3356    0.0103   -1.8550
         0         0         0         0   -1.8550    0.7093
```

We note that as expected A and B have the same eigenvalues, that R has (as per question 7) two non-zero super diagonals and that RQ is tridiagonal (using the notation from lectures, if $A_k = B = QR$ then $A_{k+1} = RQ$ is still tridiagonal).

10.
```
EDU>> n=6, A=randn(n,n), A=A+A', B=hess(A)

n =

     6


A =

   -0.4326    1.1892   -0.5883   -0.0956   -0.6918   -0.3999
   -1.6656   -0.0376    2.1832   -0.8323    0.8580    0.6900
    0.1253    0.3273   -0.1364    0.2944    1.2540    0.8156
    0.2877    0.1746    0.1139   -1.3362   -1.5937    0.7119
   -1.1465   -0.1867    1.0668    0.7143   -1.4410    1.2902
    1.1909    0.7258    0.0593    1.6236    0.5711    0.6686


A =

   -0.8651   -0.4764   -0.4630    0.1920   -1.8382    0.7910
   -0.4764   -0.0753    2.5105   -0.6577    0.6713    1.4158
   -0.4630    2.5105   -0.2728    0.4083    2.3208    0.8749
    0.1920   -0.6577    0.4083   -2.6724   -0.8794    2.3355
   -1.8382    0.6713    2.3208   -0.8794   -2.8819    1.8614
    0.7910    1.4158    0.8749    2.3355    1.8614    1.3372


B =

   -2.2139    0.7059         0         0         0         0
    0.7059   -1.3250    0.9839         0         0         0
         0    0.9839   -1.3365    2.4416         0         0
         0         0    2.4416    0.1286   -3.4362         0
         0         0         0   -3.4362   -2.0207   -3.5093
         0         0         0         0   -3.5093    1.3372

EDU>> while n > 1,...
while abs(B(n-1,n))>1.e-5,...
[Q,R]=qr(B-B(n,n)*eye(n));...
B=R*Q+B(n,n)*eye(n),...
end,lambda=B(n,n),...
n=n-1, B=B(1:n,1:n),...
end

B =

   -2.4501    0.5190         0   -0.0000         0   -0.0000
    0.5190   -1.7855    1.1970         0    0.0000    0.0000
         0    1.1970   -2.4505   -2.6811   -0.0000    0.0000
         0         0   -2.6811   -0.6336    4.4903         0
         0         0         0    4.4903    0.1045   -0.5156
```

```
         0         0         0         0   -0.5156    1.7848


B =

   -2.5656    0.4477   -0.0000    0.0000   -0.0000    0.0000
    0.4477   -2.4832    1.4646   -0.0000   -0.0000   -0.0000
         0    1.4646   -3.7071   -2.6855    0.0000   -0.0000
         0         0   -2.6855   -0.5739   -3.4520   -0.0000
         0         0         0   -3.4520    2.1220   -0.0011
         0         0         0         0   -0.0011    1.7775


B =

   -2.6561    0.4554   -0.0000    0.0000   -0.0000   -0.0000
    0.4554   -3.4430    1.7845   -0.0000    0.0000    0.0000
         0    1.7845   -4.3286   -1.7569   -0.0000    0.0000
         0         0   -1.7569    0.7788    2.9245   -0.0000
         0         0         0    2.9245    2.4409   -0.0000
         0         0         0         0   -0.0000    1.7775


lambda =

    1.7775


n =

     5


B =

   -2.6561    0.4554   -0.0000    0.0000   -0.0000
    0.4554   -3.4430    1.7845   -0.0000    0.0000
         0    1.7845   -4.3286   -1.7569   -0.0000
         0         0   -1.7569    0.7788    2.9245
         0         0         0    2.9245    2.4409


B =

   -2.7430    0.5417   -0.0000    0.0000   -0.0000
    0.5417   -4.4359    1.8190   -0.0000    0.0000
         0    1.8190   -3.8381   -0.8865    0.0000
         0         0   -0.8865    0.4324    2.4469
         0         0         0    2.4469    3.3768


B =

   -2.8514    0.7006   -0.0000    0.0000   -0.0000
    0.7006   -5.1094    1.5264   -0.0000    0.0000
         0    1.5264   -3.2219   -0.4948   -0.0000
         0         0   -0.4948   -0.6033    1.0486
         0         0         0    1.0486    4.5782


B =

   -3.0022    0.9107   -0.0000    0.0000    0.0000
    0.9107   -5.3879    1.1771   -0.0000   -0.0000
         0    1.1771   -2.8475   -0.3468    0.0000
         0         0   -0.3468   -0.7534    0.0418
         0         0         0    0.0418    4.7832


B =

   -3.2446    1.1694   -0.0000    0.0000   -0.0000
    1.1694   -5.3857    0.8712   -0.0000    0.0000
         0    0.8712   -2.6354   -0.2568   -0.0000
         0         0   -0.2568   -0.7257    0.0000
         0         0         0    0.0000    4.7835
```

```
lambda =

    4.7835


n =

     4


B =

   -3.2446    1.1694   -0.0000    0.0000
    1.1694   -5.3857    0.8712   -0.0000
         0    0.8712   -2.6354   -0.2568
         0         0   -0.2568   -0.7257


B =

   -4.5175    1.6147   -0.0000   -0.0000
    1.6147   -4.3862    0.3862    0.0000
         0    0.3862   -2.3993   -0.0057
         0         0   -0.0057   -0.6883


B =

   -5.6539    1.0905   -0.0000    0.0000
    1.0905   -3.3307    0.2265   -0.0000
         0    0.2265   -2.3184   -0.0000
         0         0   -0.0000   -0.6883


lambda =

   -0.6883


n =

     3


B =

   -5.6539    1.0905   -0.0000
    1.0905   -3.3307    0.2265
         0    0.2265   -2.3184


B =

   -6.0737    0.2061    0.0000
    0.2061   -2.9800    0.0251
         0    0.0251   -2.2494


B =

   -6.0869    0.0387   -0.0000
    0.0387   -2.9676    0.0000
         0    0.0000   -2.2485


B =

   -6.0874    0.0072    0.0000
    0.0072   -2.9672    0.0000
         0    0.0000   -2.2485


lambda =
```

```
   -2.2485


n =

     2


B =

   -6.0874    0.0072
    0.0072   -2.9672


B =

   -6.0874    0.0000
    0.0000   -2.9672


lambda =

   -2.9672


n =

     1


B =

   -6.0874

EDU>> eig(A)

ans =

   -6.0874
   -2.9672
   -2.2485
   -0.6883
    1.7775
    4.7835
```

We note that each of the eigenvalues of A (and hence B, since they're similar matrices) appears during this calculation (except for the final one which we get from the only entry of the clipped-down B matrix). Rather than copying and pasting, I've implemented the whole thing as a loop which does exactly the same thing (but involves slightly less effort on the part of the user (me!))