Numerical Solution of Differential Equations
Problem Sheet 7

Stuart Golodetz

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Disclaimer: Apologies in advance for not having finished the two final questions. I had to attempt the whole sheet this evening because of time issues this week and I haven't managed to get it all done in such a short space of time.

1. Consider the first-order hyperbolic equation $u_t + au_x + u = 0$, $-\infty < x < \infty$, $0 < t \leq T$, with a a positive constant, subject to the initial condition $u(x, 0) = u_0(x)$, $-\infty < x < \infty$, where u_0 is assumed to have a bounded second derivative on $\mathbb{R}$. This initial value problem has been approximated by the explicit finite difference scheme

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} + a\frac{U_j^n - U_{j-1}^n}{\Delta x} + U_j^n = 0, \quad j \in \mathbb{Z}, \quad n = 0, \ldots, N-1,$$

$$U_j^0 = u_0(x_j), \quad j \in \mathbb{Z},$$

where $\Delta x > 0$, $\Delta t = T/N$ and $N \geq 1$.

(a) Show that the scheme is stable in the ℓ_∞ norm provided that the Courant number $\nu = a\Delta t/\Delta x$ is less than or equal to 1.

(b) Show that the scheme is first-order accurate in the sense that its truncation error is bounded by $(M_2/2)(\Delta x + \Delta t)$ where $M_2 = \max(a \max_{x\in\mathbb{R}, 0\leq t\leq T} |u_{xx}|, \max_{x\in\mathbb{R}, 0\leq t\leq T} |u_{tt}|)$.

(c) Deduce that $\max_{0\leq n\leq N} \|U^n - u(\cdot, n\Delta t)\|_{\ell_\infty} \leq (T/2)\exp(T)M_2(\Delta x + \Delta t)$.

Answer

(a) Rearranging the scheme gives us

$$U_j^{n+1} - U_j^n + \nu(U_j^n - U_{j-1}^n) + \Delta t U_j^n = 0,$$

which in turn can be rearranged to give

$$U_j^{n+1} = (1-\nu)U_j^n + \nu U_{j-1}^n - \Delta t U_j^n.$$

Now, if we take the absolute value of this, we get:

$$\begin{aligned} |U_j^{n+1}| &= |(1-\nu)U_j^n + \nu U_{j-1}^n - \Delta t U_j^n| \\ &\leq |1-\nu||U_j^n| + |\nu||U_{j-1}^n| + \Delta t|U_j^n| \end{aligned}$$

Provided $0 \leq \nu \leq 1$, $|1-\nu| = 1-\nu$ and $|\nu| = \nu$, so we can write:

$$|U_j^{n+1}| \leq (1-\nu)|U_j^n| + \nu|U_{j-1}^n| + \Delta t|U_j^n|$$

We now observe that $\|U^n\|_{\ell_\infty} = \max_j |U_j^n|$, so we can bound all the above to give:

$$|U_j^{n+1}| \leq (1-\nu)\|U^n\|_{\ell_\infty} + \nu\|U^n\|_{\ell_\infty} + \Delta t\|U^n\|_{\ell_\infty} = (1+\Delta t)\|U^n\|_{\ell_\infty}$$

But this is true for all j, so we can rewrite the left-hand side as well, giving:

$$\|U^{n+1}\|_{\ell_\infty} \leq (1+\Delta t)\|U^n\|_{\ell_\infty}$$

It's clear by induction, therefore, that for $0 \leq n \leq N$,

$$\|U^n\|_{\ell_\infty} \leq (1+\Delta t)^n\|U^0\|_{\ell_\infty}.$$

Now, we observe that the Taylor expansion of $\exp(\Delta t)$ is:

$$\sum_{i=0}^{\infty} \frac{(\Delta t)^i}{i!} = 1 + \Delta t + \frac{1}{2}(\Delta t)^2 + \ldots$$

Thus $1 + \Delta t \leq \exp(\Delta t)$ and hence $(1 + \Delta t)^n \leq (\exp(\Delta t))^n = \exp(n\Delta t)$. We observe that $n \leq N$, so $n\Delta t \leq N\Delta t = T$. So $(1 + \Delta t)^n \leq \exp(T)$. Whence for all $0 \leq n \leq N$, we have that

$$\|U^n\|_{\ell_\infty} \leq \exp(T)\|U^0\|_{\ell_\infty},$$

provided that $0 \leq \nu \leq 1$, which is the required stability result.

(b) We first write down what the truncation error is, namely

$$T_j^n = \frac{u_j^{n+1} - u_j^n}{\Delta t} + a\frac{u_j^n - u_{j-1}^n}{\Delta x} + u_j^n,$$

where $u_j^n \equiv u(x_j, t_n)$, for $j \in \mathbb{Z}$, $0 \leq n < N$. Now, the equation we're solving gives us that $u = -u_t - au_x$, so we can rewrite the above as:

$$T_j^n = \left[\frac{u_j^{n+1} - u_j^n}{\Delta t} - u_t(x_j, t_n)\right] + a\left[\frac{u_j^n - u_{j-1}^n}{\Delta x} - u_x(x_j, t_n)\right]$$

This is looking fairly useful. We note that a Taylor expansion of u_j^{n+1} gives us

$$u_j^{n+1} = u_j^n + \Delta t u_t(x_j, t_n) + \frac{1}{2}(\Delta t)^2 u_{tt}(x_j, \tau_n)$$

for some $\tau_n \in [t_n, t_{n+1}]$. Rearranging this gives us that:

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} - u_t(x_j, t_n) = \frac{1}{2}\Delta t u_{tt}(x_j, \tau_n)$$

Furthermore, expanding u_{j-1}^n about u_j^n gives us

$$u_{j-1}^n = u_j^n - \Delta x u_x(x_j, t_n) + \frac{1}{2}(\Delta x)^2 u_{xx}(\zeta_j, t_n)$$

for some $\zeta_j \in [x_{j-1}, x_j]$. Rearranging this gives us that:

$$\frac{u_j^n - u_{j-1}^n}{\Delta x} - u_x(x_j, t_n) = -\frac{1}{2}\Delta x u_{xx}(\zeta_j, t_n)$$

So we can bound our truncation error as follows:

$$\begin{aligned}
|T_j^n| &\leq \frac{1}{2}\Delta t|u_{tt}(x_j, \tau_n)| + \frac{1}{2}a\Delta x|u_{xx}(\zeta_j, t_n)| \\
&\leq \frac{1}{2}\left(\Delta t \max_{x\in\mathbb{R}, 0\leq t\leq T} |u_{tt}| + \Delta x \cdot a \max_{x\in\mathbb{R}, 0\leq t\leq T} |u_{xx}|\right) \\
&\leq \frac{1}{2}\max\left(\max_{x\in\mathbb{R}, 0\leq t\leq T} |u_{tt}|,\ a \max_{x\in\mathbb{R}, 0\leq t\leq T} |u_{xx}|\right)(\Delta t + \Delta x) \\
&= \frac{M_2}{2}(\Delta t + \Delta x)
\end{aligned}$$

(c) If we define $e_j^n = u(x_j, t_n) - U_j^n$, then we observe that

$$\frac{e_j^{n+1} - e_j^n}{\Delta t} + a\frac{e_j^n - e_{j-1}^n}{\Delta x} + e_j^n = \frac{u_j^{n+1} - u_j^n}{\Delta t} + a\frac{u_j^n - u_{j-1}^n}{\Delta x} + u_j^n = T_j^n$$

since the U values exactly satisfy the finite difference scheme. Now, rewriting this in a similar manner to the way we went about things in part (a), we have

$$e_j^{n+1} = (1-\nu)e_j^n + \nu e_{j-1}^n - \Delta t e_j^n + \Delta t T_j^n,$$

giving us, in a similar way to before, the bound

$$\|e^{n+1}\|_{\ell_\infty} \leq (1+\Delta t)\|e^n\|_{\ell_\infty} + \Delta t\|T^n\|_{\ell_\infty}.$$

We assume that $e^0 \equiv 0$ and calculate as follows:

$$\begin{aligned}
\|e^1\|_{\ell_\infty} &\leq (1+\Delta t)\|e^0\|_{\ell_\infty} + \Delta t\|T^0\|_{\ell_\infty} = \Delta t\|T^0\|_{\ell_\infty} \\
\|e^2\|_{\ell_\infty} &\leq (1+\Delta t)\|e^1\|_{\ell_\infty} + \Delta t\|T^1\|_{\ell_\infty} = (1+\Delta t)(\Delta t\|T^0\|_{\ell_\infty}) + \Delta t\|T_1\|_{\ell_\infty} \\
\|e^3\|_{\ell_\infty} &\leq (1+\Delta t)\|e^2\|_{\ell_\infty} + \Delta t\|T_2\|_{\ell_\infty} \\
&\leq \Delta t\left[(1+\Delta t)^2\|T^0\|_{\ell_\infty} + (1+\Delta t)\|T_1\|_{\ell_\infty} + \|T_0\|_{\ell_\infty}\right] \\
&\vdots \qquad \vdots \\
\|e^n\|_{\ell_\infty} &\leq \Delta t\sum_{i=0}^{n-1}(1+\Delta t)^{n-1-i}\|T^i\|_{\ell_\infty} = \Delta t\sum_{i=1}^{n}(1+\Delta t)^{n-i}\|T^{i-1}\|_{\ell_\infty}
\end{aligned}$$

Now, since $1+\Delta t \leq \exp(\Delta t)$, as already noted, we can rewrite the above as:

$$\|e^n\|_{\ell_\infty} \leq \Delta t\sum_{i=1}^{n}(\exp(\Delta t))^{n-i}\|T^i\|_{\ell_\infty}$$

Furthermore, bounding the truncation errors above gives us:

$$\|e^n\|_{\ell_\infty} \leq \Delta t\sum_{i=1}^{n}(\exp(\Delta t))^{n-i}\max_{1\leq j\leq N}\|T^{j-1}\|_{\ell_\infty}$$

Since all the terms in the sum are positive, we can add in all the others without changing the inequality and calculate that:

$$\begin{aligned}
\|e^n\|_{\ell_\infty} &\leq \Delta t\sum_{i=1}^{N}(\exp(\Delta t))^{n-i}\max_{1\leq j\leq N}\|T^{j-1}\|_{\ell_\infty} \\
&\leq \Delta t N\exp(N\Delta t)\max_{1\leq j\leq N}\|T^{j-1}\|_{\ell_\infty}
\end{aligned}$$

This last inequality follows from the fact that each $\exp(\Delta t)^{n-i} \leq \exp(\Delta t)^N = \exp(N\Delta t)$ and that therefore each element of the sum is bounded by

$$\exp(N\Delta t)\max_{1\leq j\leq N}\|T^{j-1}\|_{\ell_\infty}.$$

Well, since $N\Delta t = T$ and $\max_{1\leq j\leq N} \|T^j\|_{\ell_\infty}$ is bounded, as was shown in the previous part, by $(M_2/2)(\Delta x + \Delta t)$, we can conclude that:

$$\|e^n\|_{\ell_\infty} \leq T\exp(T)\frac{M_2}{2}(\Delta x + \Delta t) = \frac{T}{2}\exp(T)M_2(\Delta x + \Delta t)$$

Finally, we observe that

$$\|e^n\|_{\ell_\infty} = \|u(x_j, t_n) - U_j^n\|_{\ell_\infty} = \|U_j^n - u(x_j, n\Delta t)\|_{\ell_\infty} = \|U^n - u(\cdot, n\Delta t)\|_{\ell_\infty},$$

and since our inequality on $\|e^n\|$ was true for all $0 \leq n \leq N$, we can conclude, as required, that:

$$\max_{0\leq n\leq N} \|U^n - u(\cdot, n\Delta t)\|_{\ell_\infty} \leq \frac{T}{2}\exp(T)M_2(\Delta x + \Delta t)$$

2. Consider the Lax-Friedrichs scheme

$$\frac{U_j^{n+1} - \frac{1}{2}(U_{j+1}^n + U_{j-1}^n)}{\Delta t} + a\frac{U_{j+1}^n - U_{j-1}^n}{2\Delta x} = 0, \quad j \in \mathbb{Z}, \quad n \geq 0,$$

$$U_j^0 = u_0(x_j), \quad j \in \mathbb{Z},$$

for the numerical solution of the initial value problem $u_t + au_x = 0$, $-\infty < x < \infty$, $t > 0$, subject to the initial condition $u(x, 0) = u_0(x)$, $-\infty < x < \infty$, where u_0 is assumed to have a bounded third derivative on $(-\infty, \infty)$.

(a) Explore the stability of the scheme in the ℓ_2 norm using Fourier analysis.

(b) By expanding the truncation error of the scheme into a Taylor series about a suitable point, find p and q as large as possible such that $T_j^n = O((\Delta x)^p + (\Delta x)^q)$ as $\Delta x, \Delta t \to 0$.

Answer

(a) Following the usual method for Fourier analysis (using the semidiscrete Fourier transform), we multiply through by $\Delta x e^{-ikx_j}$ and sum from $j = -\infty$ to ∞, obtaining:

$$\frac{1}{\Delta t}\left[\hat{U}^{n+1}(k) - \frac{1}{2}\left(e^{-ik\Delta x} + e^{ik\Delta x}\right)\hat{U}^n(k)\right] + \frac{a}{2\Delta x}\left[e^{-ik\Delta x} - e^{ik\Delta x}\right]\hat{U}^n(k) = 0$$

Now, multiplying through by Δt and making use of the usual identities, we deduce that:

$$\hat{U}^{n+1}(k) - \cos(k\Delta x)\hat{U}^n(k) - i\nu\sin(k\Delta x)\hat{U}^n(k) = 0$$

Hence:

$$\begin{aligned} \hat{U}^{n+1}(k) &= (\cos(k\Delta x) + i\nu\sin(k\Delta x))\hat{U}^n(k) \\ \Rightarrow \quad |\hat{U}^{n+1}(k)| &= |(\cos(k\Delta x) + i\nu\sin(k\Delta x))\hat{U}^n(k)| \\ \Rightarrow \quad |\hat{U}^{n+1}(k)| &\leq \underbrace{(\cos^2(k\Delta x) + \nu^2\sin^2(k\Delta x))}_{\phi}{}^{1/2}|\hat{U}^n(k)| \end{aligned}$$

Now, provided $\nu^2 \leq 1$ (i.e. $|\nu| \leq 1$), $\phi \leq 1$ and hence

$$|\hat{U}^{n+1}(k)| \leq |\hat{U}^n(k)|.$$

Using Parseval's identity, this becomes:

$$\|\hat{U}^{n+1}\| \leq \|\hat{U}^n\|$$

Induction on n then gives us that

$$\max_{0\leq n\leq N} \|U^n\|_{\ell_2} \leq \|U^0\|_{\ell_2},$$

which gives us stability, provided (as already noted) that $|\nu| \leq 1$.

(b) First we write down the truncation error for the scheme, namely

$$T_j^n = \frac{u_j^{n+1} - \frac{1}{2}(u_{j+1}^n + u_{j-1}^n)}{\Delta t} + a\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x},$$

where $u_j^n \equiv u(x_j, t_n)$ as in question 1. Now let's do our Taylor expansion about the point (x_j, t_n). We calculate that

$$u(x_j, t_{n+1}) = u(x_j, t_n) + \Delta t u_t(x_j, t_n) + \frac{1}{2}(\Delta t)^2 u_{tt}(x_j, \tau_n)$$

for some $\tau_n \in [t_n, t_{n+1}]$. Furthermore,

$$u(x_{j+1}, t_n) = u(x_j, t_n) + \Delta x u_x(x_j, t_n) + \frac{1}{2}(\Delta x)^2 u_{xx}(x_j, \zeta_n)$$

for some $\zeta_n \in [x_j, x_{j+1}]$, and

$$u(x_{j-1}, t_n) = u(x_j, t_n) - \Delta x u_x(x_j, t_n) + \frac{1}{2}(\Delta x)^2 u_{xx}(x_j, \chi_n)$$

for some $\chi_n \in [x_{j-1}, x_j]$. So

$$u(x_{j+1}, t_n) + u(x_{j-1}, t_n) = 2u(x_j, t_n) + \frac{1}{2}(\Delta x)^2 \left[u_{xx}(x_j, \zeta_n) + u_{xx}(x_j, \chi_n)\right]$$

and

$$u(x_{j+1}, t_n) - u(x_{j-1}, t_n) = 2\Delta x u_x(x_j, t_n) + \frac{1}{2}(\Delta x)^2 \left[u_{xx}(x_j, \zeta_n) - u_{xx}(x_j, \chi_n)\right].$$

Plugging all this into our formula for the truncation error, we get:

$$\begin{aligned}
T_j^n &= \frac{1}{\Delta t}\left[\Delta t u_t(x_j, t_n) + \frac{1}{2}(\Delta t)^2 u_{tt}(x_j, \tau_n) - \frac{1}{4}(\Delta x)^2 \left(u_{xx}(x_j, \zeta_n) + u_{xx}(x_j, \chi_n)\right)\right] \\
&+ \frac{a}{2\Delta x}\left[2\Delta x u_x(x_j, t_n) + \frac{1}{2}(\Delta x)^2 \left(u_{xx}(x_j, \zeta_n) - u_{xx}(x_j, \chi_n)\right)\right] \\
&= u_t(x_j, t_n) + \frac{1}{2}\Delta t u_{tt}(x_j, \tau_n) - \frac{(\Delta x)^2}{4\Delta t}\left(u_{xx}(x_j, \zeta_n) + u_{xx}(x_j, \chi_n)\right) \\
&+ a u_x(x_j, t_n) + \frac{a}{4}\Delta x \left(u_{xx}(x_j, \zeta_n) - u_{xx}(x_j, \chi_n)\right) \\
&= \frac{1}{2}\Delta t u_{tt}(x_j, \tau_n) - \frac{(\Delta x)^2}{4\Delta t}\left(u_{xx}(x_j, \zeta_n) + u_{xx}(x_j, \chi_n)\right) \\
&+ \frac{a}{4}\Delta x \left(u_{xx}(x_j, \zeta_n) - u_{xx}(x_j, \chi_n)\right)
\end{aligned}$$

Since $1/\Delta t = a/(\nu \Delta x)$, this becomes:

$$\begin{aligned}
T_j^n &= \frac{1}{2}\Delta t u_{tt}(x_j, \tau_n) - \frac{a\Delta x}{4\nu}\left(u_{xx}(x_j, \zeta_n) + u_{xx}(x_j, \chi_n)\right) \\
&+ \frac{a}{4}\Delta x \left(u_{xx}(x_j, \zeta_n) - u_{xx}(x_j, \chi_n)\right)
\end{aligned}$$

From this it's clear that the truncation error is in $O(\Delta x + \Delta t)$, so $p = q = 1$.

3. Sketch the characteristic curves for the equation $u_t + a(x)u_x = 0$ for $0 \leq x \leq 1$, $t \geq 0$, when $a(x) = x-(1/2)$. Set up the upwind scheme for the numerical solution of this equation, subject to the initial condition $u(x,0) = u_0(x)$, on a uniform mesh $\{x_j = j\Delta x,\ j = 0, 1, \ldots, J\}$ and constant time step $\Delta t = T/N$ where $N \geq 1$ and $T > 0$ is the final time. Explain carefully why no boundary conditions are needed at $x = 0$ or $x = 1$.

   Derive a bound on the error between u and its upwind finite difference approximation U in the ℓ_∞ norm on the interval $[0, 1]$, considering both even and odd J.

   Assuming that $u_0(x) = x(1-x)$, obtain an explicit bound on the error between u and U by estimating the derivatives in the truncation error.

Answer

Before we can sketch the characteristic curves, we need to work out what they are. We observe that if we denote a characteristic curve as $(x(s), t(s))$, we can find a function $u(x(s), t(s))$ which is constant along the curve by choosing $dx/ds = a(x)$ and $dt/ds = 1$. In that case:

$$\frac{d}{ds}u(x(s), t(s)) = \frac{dx}{ds}u_x + \frac{dt}{ds}u_t = a(x)u_x + u_t = 0$$

So to find our characteristic curves, we need to solve the two ODEs $dx/ds = a(x)$ and $dt/ds = 1$, subject to the initial conditions $t(0) = t_0$ and $x(0) = x_0$. Well, the first one is:

$$dx/ds = x - \frac{1}{2}, \quad x(0) = x_0$$

By separation of variables, we get:

$$\int_{x_0}^{x} \frac{1}{x - \frac{1}{2}} dx = \int_0^s ds$$

Whence:

$$\left[\ln\left(x - \frac{1}{2}\right)\right]_{x_0}^{x} = \ln\left(\frac{x - \frac{1}{2}}{x_0 - \frac{1}{2}}\right) = s$$

Rearranging this gives:

$$x = \frac{1}{2} + \left(x_0 - \frac{1}{2}\right)e^s$$

The second ODE can be solved by writing:

$$\int_{t_0}^{t} dt = \int_0^s ds$$

Whence:

$$[t]_{t_0}^{t} = t - t_0 = s \quad \Leftrightarrow \quad t = t_0 + s$$

Assuming $t_0 = 0$, this gives us characteristic curves defined by:

$$x = \frac{1}{2} + \left(x_0 - \frac{1}{2}\right)e^t$$

These can be plotted in the (x, t) plane (see separate sheet).

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As far as formulating the upwind scheme for this goes, we know by analogy with the lecture notes that where $a(x) < 0$, the scheme is

$$\frac{U_j^{m+1} - U_j^m}{\Delta t} + a(x_j)\frac{U_{j+1}^m - U_j^m}{\Delta x} = 0,$$

and where $a(x) > 0$, the scheme is

$$\frac{U_j^{m+1} - U_j^m}{\Delta t} + a(x_j)\frac{U_j^m - U_{j-1}^m}{\Delta x} = 0.$$

Rewriting these using the notation $\nu_j = (x_j - \frac{1}{2})\frac{\Delta t}{\Delta x} = a(x_j)\frac{\Delta t}{\Delta x}$, the first one becomes

$$U_j^{m+1} = (1 + \nu_j)U_j^m - \nu_j U_{j+1}^m$$

and the second one becomes

$$U_j^{m+1} = (1 - \nu_j)U_j^m + \nu_j U_{j-1}^m.$$

Now, the question becomes: when is $a(x) < 0$ and when is it > 0? Let's consider what happens when J is odd. In that case, $x_0 = 0$, $x_1 = \Delta x = 1/J$, $x_2 = 2/J$, ..., $x_J = J/J = 1$. Then $a(x_j) < 0$ provided $x_j < \frac{1}{2}$, i.e. for $j = 0, 1, \dots, \frac{1}{2}(J - 1)$. Correspondingly, $a(x_j) > 0$ for $j = \frac{1}{2}(J - 1) + 1, \dots, J$. So when J is odd, our upwind scheme is:

$$U_j^{m+1} = \begin{cases} (1 + \nu_j)U_j^m - \nu_j U_{j+1}^m, & j = 0, 1, \dots, \frac{1}{2}(J - 1) \\ (1 - \nu_j)U_j^m + \nu_j U_{j-1}^m, & j = \frac{1}{2}(J - 1) + 1, \dots, J \end{cases}$$

What about when J is even? Then $a(x_{J/2}) = 0$, whence $U_{J/2}^{n+1} = U_{J/2}^n$. So for all n, $U_{J/2}^n = u_0(1/2)$ by induction. So our scheme is:

$$U_j^{m+1} = \begin{cases} (1 + \nu_j)U_j^m - \nu_j U_{j+1}^m, & j = 0, 1, \dots, \frac{J}{2} - 1 \\ u_0(1/2), & j = \frac{J}{2} \\ (1 - \nu_j)U_j^m + \nu_j U_{j-1}^m, & j = \frac{J}{2} + 1, \dots, J \end{cases}$$

So why don't we need boundary conditions at $x = 0$ and $x = 1$? Well, consider that in both the odd and even schemes, the formula for U_0^{m+1} only involves U_0^m and U_1^m. The formula which has a U_{j-1}^m term in it doesn't apply when $j = 0$! Similarly, the formula for U_J^{m+1} only involves U_J^m and U_{J-1}^m.

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For $a(x) < 0$, the truncation error is defined as:

$$\frac{u_j^{m+1} - u_j^m}{\Delta t} + \left(x_j - \frac{1}{2}\right)\frac{u_{j+1}^m - u_j^m}{\Delta x}$$

For $a(x) > 0$, it is, by contrast:

$$\frac{u_j^{m+1} - u_j^m}{\Delta t} + \left(x_j - \frac{1}{2}\right)\frac{u_j^m - u_{j-1}^m}{\Delta x}$$

Consider doing a Taylor expansion around (x_j, t_m). Then:

$$u_j^{m+1} = u + \Delta t u_t + \frac{1}{2}(\Delta t)^2 u_{tt} + \dots$$

Furthermore:

$$u_{j\pm1}^m = u \pm \Delta x u_x + \frac{1}{2}(\Delta x)^2 u_{xx} + \dots$$

So the truncation error when $a(x) < 0$ is:

$$\begin{aligned} T_j^n &\approx u_t + \frac{1}{2}\Delta t u_{tt} + \left(x_j - \frac{1}{2}\right)\left(u_x + \frac{1}{2}\Delta x u_{xx}\right) \\ &= \frac{1}{2}\Delta t u_{tt} + \frac{1}{2}\Delta x\left(x_j - \frac{1}{2}\right)u_{xx} \end{aligned}$$

Similarly, the truncation error when $a(x) > 0$ is:

$$\begin{aligned} T_j^n &\approx u_t + \frac{1}{2}\Delta t u_{tt} + \left(x_j - \frac{1}{2}\right)\left(u_x - \frac{1}{2}\Delta x u_{xx}\right) \\ &= \frac{1}{2}\Delta t u_{tt} - \frac{1}{2}\Delta x\left(x_j - \frac{1}{2}\right)u_{xx} \end{aligned}$$

So for the odd scheme:

$$T_j^n = \begin{cases} \frac{1}{2}\Delta t u_{tt} + \frac{1}{2}\Delta x\left(x_j - \frac{1}{2}\right)u_{xx}, & j = 0, 1, \dots, \frac{1}{2}(J-1) \\ \frac{1}{2}\Delta t u_{tt} - \frac{1}{2}\Delta x\left(x_j - \frac{1}{2}\right)u_{xx}, & j = \frac{1}{2}(J-1) + 1, \dots, J \end{cases}$$

And for the even scheme:

$$T_j^n = \begin{cases} \frac{1}{2}\Delta t u_{tt} + \frac{1}{2}\Delta x\left(x_j - \frac{1}{2}\right)u_{xx}, & j = 0, 1, \dots, \frac{J}{2} - 1 \\ 0, & j = \frac{J}{2} \\ \frac{1}{2}\Delta t u_{tt} - \frac{1}{2}\Delta x\left(x_j - \frac{1}{2}\right)u_{xx}, & j = \frac{J}{2} + 1, \dots, J \end{cases}$$

TODO

4. Verify that the function u defined implicitly by the equation

$$u(x,t) = u_0(x - u(x,t)t)$$

is a solution of the initial value problem

$$u_t + uu_x = 0, \quad -\infty < x < \infty, \quad t > 0; \quad u(x,0) = u_0(x), \quad -\infty < x < \infty, \qquad (1)$$

provided that u is a differentiable function of x and t. Given that $x_0 \in \mathbb{R}$, show further that $u(x,t)$ has the constant value $u_0(x_0)$ along the straight line $\{(x,t) \ : \ x - x_0 = tu_0(x_0)\}$.

Assuming that $\epsilon > 0$ is a small real number, show that the lines through the points $(x_0, 0)$ and $(x_0 + \epsilon, 0)$ meet at a point, whose limit as $\epsilon \to 0$ is

$$\left(x_0 - \frac{u_0(x_0)}{u_0'(x_0)}, \ -\frac{1}{u_0'(x_0)} \right).$$

Deduce that if $u_0'(x) \geq 0$ for all x then $u(x,t)$ is well-defined for all $x \in \mathbb{R}$ and all $t > 0$. More generally, show that if $u_0'(x)$ takes negative values, the solution $u(x,t)$ is well-defined for all $x \in \mathbb{R}$ and all $t \in [0, t_c]$, where $t_c = -1/M$ and M is the largest negative value of $u_0'(x)$ for $x \in \mathbb{R}$.

Show that for $u_0(x) = \exp[-10(4x-1)^2]$ the critical value t_c is equal to $\exp((1/2))/(8\sqrt{5})$, which is about 0.092.

Formulate the (first-order) upwind scheme for the numerical solution of the initial value problem (1). Show that if $(\Delta t/\Delta x)\|u_0\|_{L_\infty(\mathbb{R})} \leq 1$, then $\max_{n \geq 0} \|U^n\|_{\ell_\infty} \leq \|U^0\|_{\ell_\infty}$.

Answer

For the given function u, we calculate that

$$\begin{aligned}
u_x(x,t) &= \frac{\partial}{\partial x} u_0(x - u(x,t)t) \\
&= u_0'(x - u(x,t)t) \times \frac{\partial}{\partial x}(x - u(x,t)t) \\
&= u_0'(x - u(x,t)t) \times (1 - u_x(x,t)t) \\
&= u_0'(x - u(x,t)t) - u_x(x,t)[t \cdot u_0'(x - u(x,t)t)]
\end{aligned}$$

Thus:

$$u_x(x,t) = \frac{u_0'(x - u(x,t)t)}{1 + t \cdot u_0'(x - u(x,t)t)}$$

Furthermore:

$$\begin{aligned}
u_t(x,t) &= \frac{\partial}{\partial t} u_0(x - u(x,t)t) \\
&= u_0'(x - u(x,t)t) \times \frac{\partial}{\partial t}(x - u(x,t)t) \\
&= u_0'(x - u(x,t)t) \times (-[u_t(x,t)t + u(x,t)]) \\
&= -u_0'(x - u(x,t)t) \cdot u(x,t) - u_t(x,t)[t \cdot u_0'(x - u(x,t)t)]
\end{aligned}$$

So:

$$u_t(x,t) = \frac{-u_0'(x - u(x,t)t) \cdot u(x,t)}{1 + t \cdot u_0'(x - u(x,t)t)}$$

Plugging these into the initial value problem, we get

$$\begin{aligned} u_t + uu_x &= \frac{-u_0'(x-u(x,t)t)\cdot u(x,t)}{1+t\cdot u_0'(x-u(x,t)t)} + u(x,t)\frac{u_0'(x-u(x,t)t)}{1+t\cdot u_0'(x-u(x,t)t)} \\ &= 0 \end{aligned}$$

as required. So the given u is a solution of the initial value problem.

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Take an arbitrary point (x,t) along the straight line given. This x and t clearly satisfy

$$x = tu_0(x_0) + x_0.$$

We calculate that

$$\begin{aligned} u(x,t) &= u_0(x-u(x,t)t) \\ &= u_0(tu_0(x_0)+x_0-u(x,t)t) \\ &= u_0(t[u_0(x_0)-u(x,t)]+x_0) \end{aligned}$$

Now if $u(x,t) = u_0(x_0)$ then clearly the above equation becomes:

$$u_0(x_0) = u_0(t\cdot 0 + x_0) = u_0(x_0).$$

In other words, if $u(x,t) = u_0(x_0)$ for an arbitrary point on the straight line, then the whole thing works as expected.

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The line through $(x_0, 0)$ is

$$x = x_0 + tu_0(x_0)$$

and that through $(x_0+\epsilon, 0)$ is

$$x = x_0 + \epsilon + tu_0(x_0+\epsilon).$$

These meet when

$$\begin{aligned} & tu_0(x_0) = \epsilon + tu_0(x_0+\epsilon) \\ \Leftrightarrow \quad & -\epsilon = t(u_0(x_0+\epsilon) - u_0(x_0)) \\ \Leftrightarrow \quad & \frac{u_0(x_0+\epsilon)-u_0(x_0)}{\epsilon} = -\frac{1}{t}. \end{aligned}$$

In the limit, as $\epsilon \longrightarrow 0$, this condition becomes:

$$u_0'(x_0) = -\frac{1}{t}$$

In other words, the lines meet when

$$t = -\frac{1}{u_0'(x_0)},$$

which, since $x = x_0 + tu_0(x_0)$, is when

$$x = x_0 - \frac{u_0(x_0)}{u_0'(x_0)}.$$

So in the limit, as $\epsilon \longrightarrow 0$, the lines meet, as required, at the point

$$\left(x_0 - \frac{u_0(x_0)}{u_0'(x_0)}, -\frac{1}{u_0'(x_0)}\right).$$

Incidentally, is it just coincidence that this looks a lot like the thing from the Newton-Raphson method?

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TODO

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If $u_0(x) = \exp[-10(4x-1)^2]$, then we calculate that

$$u_0'(x) = -20(4x-1) \cdot 4 \cdot \exp[-10(4x-1)^2] = -80(4x-1)\exp(-10(4x-1)^2)$$

We want the largest negative value of this, so differentiate it again and equate the second derivative to 0:

$$\begin{aligned} u_0''(x) &= -80\left[(4x-1)\cdot -80(4x-1)\exp(-10(4x-1)^2) + 4\exp(-10(4x-1)^2)\right] \\ &= 80\exp(-10(4x-1)^2)\left[80(4x-1)^2 - 4\right] \\ &= 0 \end{aligned}$$

Neither the constant 80 nor the exponential can be zero, so this becomes:

$$\begin{aligned} & 80(4x-1)^2 - 4 = 0 \\ \Leftrightarrow \quad & (4x-1)^2 = \frac{1}{20} \\ \Leftrightarrow \quad & x = \frac{1}{4}\left[1 \pm \sqrt{\frac{1}{20}}\right] \end{aligned}$$

From a sketch of $u_0'(x)$, we see that the root we want is the $+$ one. When $x = \frac{1}{4}\left[1 + \sqrt{\frac{1}{20}}\right]$,

$$\begin{aligned} u_0'(x) &= -80\left(\sqrt{\frac{1}{20}}\right)\exp\left(-10\cdot\frac{1}{20}\right) \\ &= -\underbrace{\sqrt{320}}_{8\sqrt{5}}\exp\left(-\frac{1}{2}\right) \end{aligned}$$

So, as required:

$$t_c = \frac{-1}{\left(-8\sqrt{5}\exp\left(-\frac{1}{2}\right)\right)} = \frac{\exp\left(\frac{1}{2}\right)}{8\sqrt{5}}$$

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TODO