Numerical Solution of Differential Equations
Problem Sheet 6

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1. The diffusion equation $u_t = u_{xx}$, $-\infty < x < \infty$, subject to the initial condition $u(x, 0) = u_0(x)$, $-\infty < x < \infty$, is approximated by the finite difference scheme (*Crandall's scheme*):

$$U_j^{n+1} - \frac{1}{2}(\nu - \zeta)(U_{j+1}^{n+1} - 2U_j^{n+1} + U_{j-1}^{n+1}) = U_j^n + \frac{1}{2}(\nu + \zeta)(U_{j+1}^n - 2U_j^n + U_{j-1}^n)$$

with $U_j^0 = u_0(x_j)$, where $\Delta t > 0$, $\Delta x > 0$, $\nu = \Delta t/(\Delta x)^2$ and ζ is a fixed constant. Show that if ν is a fixed real number, then the truncation error, T_j^n, obeys

$$T_j^n = \begin{cases} O((\Delta x)^2) \text{ if } \zeta \neq 1/6 \\ O((\Delta x)^4) \text{ if } \zeta = 1/6. \end{cases}$$

Answer

Let's start by defining the truncation error:

$$T_j^n = u_j^{n+1} - u_j^n - \frac{1}{2}(\nu - \zeta)(u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}) - \frac{1}{2}(\nu + \zeta)(u_{j+1}^n - 2u_j^n + u_{j-1}^n)$$

There's no obvious place at which to do our Taylor expansion here, so we'll choose to expand around (x_j, t_n) because it simplifies at least some of the calculations.[1] So, let's start by noting that:

$$u_{j\pm1}^n = \left[u \pm \Delta x u_x + \frac{1}{2}(\Delta x)^2 u_{xx} \pm \frac{1}{6}(\Delta x)^3 u_{xxx} + \frac{1}{24}(\Delta x)^4 u_{xxxx} + \dots\right]_j^n$$

Thus:

$$u_{j+1}^n - 2u_j^n + u_{j-1}^n = \left[(\Delta x)^2 u_{xx} + \frac{1}{12}(\Delta x)^4 u_{xxxx} + \frac{2}{6!}(\Delta x)^6 u_{xxxxxx} + \dots\right]_j^n$$

Furthermore:

$$u_j^{n+1} = \left[u + \Delta t u_t + \frac{1}{2}(\Delta t)^2 u_{tt} + \frac{1}{6}(\Delta t)^3 u_{ttt} + \frac{1}{24}(\Delta t)^4 u_{tttt} + \dots\right]_j^n$$

Now we need to do some 2D Taylor expansions. Accordingly, we recall (as we did last week) that the Taylor series expansion of a real function of two variables is given by:

$$f(x + \delta x, y + \delta y) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{(\delta x)^j}{j!}\frac{(\delta y)^k}{k!}\frac{\partial^j}{\partial x^j}\frac{\partial^k}{\partial y^k} f(x, y)$$

We hence calculate that:

$$\begin{aligned}
u_{j\pm1}^{n+1} &= [u \pm \Delta x u_x + \Delta t u_t \\
&+ \frac{1}{2}\left((\Delta x)^2 u_{xx} \pm 2\Delta x \Delta t u_{xt} + (\Delta t)^2 u_{tt}\right) \\
&+ \frac{1}{6}\left(\pm(\Delta x)^3 u_{xxx} + 3(\Delta x)^2 \Delta t u_{xxt} \pm 3\Delta x(\Delta t)^2 u_{xtt} + (\Delta t)^3 u_{ttt}\right) \\
&+ \frac{1}{24}\left((\Delta x)^4 u_{xxxx} \pm 4(\Delta x)^3 \Delta t u_{xxxt} + 6(\Delta x)^2(\Delta t)^2 u_{xxtt} \pm 4\Delta x(\Delta t)^3 u_{xttt} + (\Delta t)^4 u_{tttt}\right) \\
&+ \dots]_j^n
\end{aligned}$$

[1]The alternative of $(x_j, t_{n+1/2})$ seems less appealing after using that for last week's sheet.

Thus:

$$
\begin{aligned}
u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1} &= \left[2u + 2\Delta t u_t + (\Delta x)^2 u_{xx} + (\Delta t)^2 u_{tt}\right]_j^n \\
&+ \left[(\Delta x)^2 \Delta t u_{xxt} + \frac{1}{3}(\Delta t)^3 u_{ttt}\right]_j^n \\
&+ \left[\frac{1}{12}(\Delta x)^4 u_{xxxx} + \frac{1}{2}(\Delta x)^2(\Delta t)^2 u_{xxtt} + \frac{1}{12}(\Delta t)^4 u_{tttt}\right]_j^n \\
&+ \left[\frac{10}{5!}(\Delta x)^4 \Delta t u_{xxxxt} + \frac{20}{5!}(\Delta x)^2(\Delta t)^3 u_{xxttt} + \frac{2}{5!}(\Delta t)^5 u_{ttttt}\right]_j^n \\
&+ \left[\frac{2}{6!}(\Delta x)^6 u_{xxxxxx} + \frac{30}{6!}(\Delta x)^4(\Delta t)^2 u_{xxxxtt}\right]_j^n \\
&+ \left[\frac{30}{6!}(\Delta x)^2(\Delta t)^4 u_{xxtttt} + \frac{2}{6!}(\Delta t)^6 u_{tttttt}\right]_j^n \\
&+ \ldots \\
&- 2\left[u + \Delta t u_t + \frac{1}{2}(\Delta t)^2 u_{tt} + \frac{1}{6}(\Delta t)^3 u_{ttt} + \frac{1}{24}(\Delta t)^4 u_{tttt} + \ldots\right]_j^n \\
&= \left[(\Delta x)^2 u_{xx} + (\Delta x)^2 \Delta t u_{xxt} + \frac{1}{12}(\Delta x)^4 u_{xxxx} + \frac{1}{2}(\Delta x)^2(\Delta t)^2 u_{xxtt}\right]_j^n \\
&+ \left[\frac{1}{12}(\Delta x)^4 \Delta t u_{xxxxt} + \frac{1}{6}(\Delta x)^2(\Delta t)^3 u_{xxttt}\right]_j^n \\
&+ \left[\frac{2}{6!}(\Delta x)^6 u_{xxxxxx} + \frac{1}{24}(\Delta x)^4(\Delta t)^2 u_{xxxxtt}\right]_j^n \\
&+ \left[\frac{1}{24}(\Delta x)^2(\Delta t)^4 u_{xxtttt} + \ldots\right]_j^n
\end{aligned}
$$

Now, since ν is a fixed real number, we note that $\Delta t = \nu(\Delta x)^2 \in O((\Delta x)^2)$. Furthermore, in our formula for the truncation error, the above is multiplied by $-\frac{1}{2}(\nu - \zeta)$, which equals $-\frac{1}{2}(\Delta t/(\Delta x)^2 - \zeta)$. Given that the question suggests that we end up with something which is either $O((\Delta x)^2)$ or $O((\Delta x)^4)$, we know that we won't have to worry about terms in the above of order greater than $(\Delta x)^6$, so we can simplify things immensely by eliding them at this stage to give:

$$
\begin{aligned}
u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1} &= \left[(\Delta x)^2 u_{xx} + \nu(\Delta x)^4 u_{xxt} + \frac{1}{12}(\Delta x)^4 u_{xxxx} + \frac{1}{2}\nu^2(\Delta x)^6 u_{xxtt}\right]_j^n \\
&+ \left[\frac{1}{12}\nu(\Delta x)^6 u_{xxxxt} + \frac{2}{6!}(\Delta x)^6 u_{xxxxxx} + \ldots\right]_j^n
\end{aligned}
$$

Plugging all this into our formula for the truncation error, we get:

$$
\begin{aligned}
T_j^n \approx\ & \left[\Delta t u_t + \frac{1}{2}(\Delta t)^2 u_{tt} + \frac{1}{6}(\Delta t)^3 u_{ttt}\right]_j^n \\
-\ & \frac{1}{2}(\nu-\zeta)\left[(\Delta x)^2 u_{xx} + \nu(\Delta x)^4 u_{xxt} + \frac{1}{12}(\Delta x)^4 u_{xxxx} + \frac{1}{2}\nu^2(\Delta x)^6 u_{xxtt}\right]_j^n \\
-\ & \frac{1}{2}(\nu-\zeta)\left[\frac{1}{12}\nu(\Delta x)^6 u_{xxxxt} + \frac{2}{6!}(\Delta x)^6 u_{xxxxxx}\right]_j^n \\
-\ & \frac{1}{2}(\nu+\zeta)\left[(\Delta x)^2 u_{xx} + \frac{1}{12}(\Delta x)^4 u_{xxxx}\right]_j^n \\
=\ & \left[\nu(\Delta x)^2 u_t + \frac{1}{2}\nu^2(\Delta x)^4 u_{tt} + \frac{1}{6}\nu^3(\Delta x)^6 u_{ttt}\right]_j^n \\
-\ & \frac{1}{2}(\nu-\zeta)\left[(\Delta x)^2 u_{xx} + \nu(\Delta x)^4 u_{xxt} + \frac{1}{12}(\Delta x)^4 u_{xxxx} + \frac{1}{2}\nu^2(\Delta x)^6 u_{xxtt}\right]_j^n \\
-\ & \frac{1}{2}(\nu-\zeta)\left[\frac{1}{12}\nu(\Delta x)^6 u_{xxxxt} + \frac{2}{6!}(\Delta x)^6 u_{xxxxxx}\right]_j^n \\
-\ & \frac{1}{2}(\nu+\zeta)\left[(\Delta x)^2 u_{xx} + \frac{1}{12}(\Delta x)^4 u_{xxxx}\right]_j^n \\
=\ & (\Delta x)^2\left[\nu u_t - \frac{1}{2}(\nu-\zeta)u_{xx} - \frac{1}{2}(\nu+\zeta)u_{xx}\right]_j^n \\
+\ & (\Delta x)^4\left[\frac{1}{2}\nu^2 u_{tt} - \frac{1}{2}(\nu-\zeta)\left(\nu u_{xxt} + \frac{1}{12}u_{xxxx}\right) - \frac{1}{2}(\nu+\zeta)\left(\frac{1}{12}u_{xxxx}\right)\right]_j^n \\
+\ & (\Delta x)^6\left[\frac{1}{6}\nu^3 u_{ttt} - \frac{1}{2}(\nu-\zeta)\left(\frac{1}{2}\nu^2 u_{xxtt} + \frac{1}{12}\nu u_{xxxxt} + \frac{2}{6!}u_{xxxxxx}\right)\right]_j^n \\
=\ & (\Delta x)^2\left[\nu(\underbrace{u_t - u_{xx}}_{=0})\right]_j^n \\
+\ & \frac{1}{2}(\Delta x)^4\left[\nu^2 u_{tt} - \frac{1}{6}\nu u_{xxxx} - \nu^2 u_{xxt} + \zeta\nu u_{xxt}\right]_j^n \\
+\ & \frac{1}{24}(\Delta x)^6\Big[4\nu^3 u_{ttt} + 6\zeta\nu^2 u_{xxtt} - 6\nu^3 u_{xxtt} + \zeta\nu u_{xxxxt} - \nu^2 u_{xxxxt} \\
& \qquad + \frac{1}{30}\zeta u_{xxxxxx} - \frac{1}{30}\nu u_{xxxxxx}\Big]_j^n \\
=\ & \frac{1}{2}(\Delta x)^4\left[(\nu^2(\underbrace{u_t - u_{xx}}_{=0}))_t - \frac{1}{6}\nu u_{xxxx} + \zeta\nu u_{xxt}\right]_j^n \\
+\ & \frac{1}{24}(\Delta x)^6\Big[4\nu^3 u_{ttt} + 6\zeta\nu^2 u_{xxtt} - 6\nu^3 u_{xxtt} + \zeta\nu u_{xxxxt} - \nu^2 u_{xxxxt} \\
& \qquad + \frac{1}{30}\zeta u_{xxxxxx} - \frac{1}{30}\nu u_{xxxxxx}\Big]_j^n
\end{aligned}
$$

So we get:

$$
\begin{aligned}
T_j^n &\approx \frac{1}{2}\nu(\Delta x)^4\left[\zeta u_{xxt} - \frac{1}{6}u_{xxxx}\right]_j^n \\
&+ \frac{1}{24}(\Delta x)^6\Big[4\nu^3 u_{ttt} + 6\zeta\nu^2 u_{xxtt} - 6\nu^3 u_{xxtt} + \zeta\nu u_{xxxxt} - \nu^2 u_{xxxxt} \\
&\qquad\qquad + \frac{1}{30}\zeta u_{xxxxxx} - \frac{1}{30}\nu u_{xxxxxx}\Big]_j^n \\
&= \frac{1}{2}\Delta t(\Delta x)^2\left[\zeta u_{xxt} - \frac{1}{6}u_{xxxx}\right]_j^n \\
&+ \Big[\frac{1}{6}(\Delta t)^3 u_{ttt} + \frac{1}{4}(\Delta t)^2(\Delta x)^2\zeta u_{xxtt} - \frac{1}{4}(\Delta t)^3 u_{xxtt} + \frac{1}{4!}\zeta\Delta t(\Delta x)^4 u_{xxxxt} \\
&\qquad - \frac{1}{4!}(\Delta t)^2(\Delta x)^2 u_{xxxxt} + \frac{1}{6!}\zeta(\Delta x)^6 u_{xxxxxx} - \frac{1}{6!}\Delta t(\Delta x)^4 u_{xxxxxx}\Big]_j^n \\
&= (\Delta x)^2\left[\frac{1}{2}\Delta t\zeta u_{xxt} - \frac{1}{12}\Delta t u_{xxxx} + \frac{1}{4}(\Delta t)^2\zeta u_{xxtt} - \frac{1}{4!}(\Delta t)^2 u_{xxxxt}\right]_j^n \\
&+ O((\Delta x)^4)
\end{aligned}
$$

Now, the coefficient of $(\Delta x)^2$ is clearly non-zero in general, but when $\zeta = \frac{1}{6}$, we have:

$$
\begin{aligned}
&\left[\frac{1}{12}\Delta t u_{xxt} - \frac{1}{12}\Delta t u_{xxxx} + \frac{1}{4!}(\Delta t)^2 u_{xxtt} - \frac{1}{4!}(\Delta t)^2 u_{xxxxt}\right]_j^n \\
= &\left[\left(\frac{1}{12}\Delta t(\underbrace{u_t - u_{xx}}_{=0})\right)_{xx} + \left(\frac{1}{4!}(\Delta t)^2(\underbrace{u_t - u_{xx}}_{=0})\right)_{xxt}\right]_j^n \\
= &\ 0
\end{aligned}
$$

Thus when $\zeta = \frac{1}{6}$, the truncation error is $O((\Delta x)^4)$, otherwise it is $O((\Delta x)^2)$, as required.

2. Letting $\nu = \Delta t/(\Delta x)^2$, $\Delta x = 1/J$, $J \geq 2$, $\Delta t = T/N$, $N \geq 1$, $T > 0$, consider the θ-scheme

$$U_j^{n+1} - U_j^n = \nu \left[\theta(U_{j+1}^{n+1} - 2U_j^{n+1} + U_{j-1}^{n+1}) + (1-\theta)(U_{j+1}^n - 2U_j^n + U_{j-1}^n)\right],$$

where $j = 0, 1, \ldots, J-1$, $0 \leq n \leq N-1$, with $0 \leq \theta \leq 1$,

$$U_0^n = 0, \quad U_J^n = 0, \quad 0 \leq n \leq N-1,$$

and

$$U_j^0 = u_0(x_j), \quad 1 \leq j \leq J-1,$$

for the numerical solution of the initial boundary value problem $u_t = u_{xx}$, $0 < x < 1$, $0 < t \leq T$, subject to homogeneous Dirichlet boundary conditions at $x = 0$ and $x = 1$, and the initial condition $u(x,0) = u_0(x)$ $0 < x < 1$.

Show that if $2\nu(1-\theta) \leq 1$, then the θ-scheme obeys the following maximum principle:

$$U_{\min}^n \leq U_j^n \leq U_{\max}^n,$$

where

$$U_{\min}^n = \min\left\{U_0^m,\ 0 \leq m \leq n;\ U_j^0,\ 0 \leq j \leq J;\ U_J^m,\ 0 \leq m \leq n\right\},$$

and

$$U_{\max}^n = \max\left\{U_0^m,\ 0 \leq m \leq n;\ U_j^0,\ 0 \leq j \leq J;\ U_J^m,\ 0 \leq m \leq n\right\}.$$

Answer

As per the proof of the *Discrete Maximum Principle* in the notes, we start by rewriting the θ-method as:

$$(1+2\theta\nu)U_j^{n+1} = \theta\nu\left(U_{j+1}^{n+1} + U_{j-1}^{n+1}\right) + (1-\theta)\nu\left(U_{j+1}^n + U_{j-1}^n\right) + [1 - 2(1-\theta)\nu]U_j^n$$

Consider the proof for the maximum value (the proof for the minimum value is analogous). Suppose that for $(x,t) \in [0,J] \times [0, n+1]$, U attains its maximum value at an internal grid point U_j^{m+1}, $1 \leq j \leq J-1$, $0 \leq m \leq n$. (If this is not the case, the proof is complete, since then the maximum value in the range must be on the boundary.) Now, define:

$$U^\star = \max\left\{U_{j+1}^{m+1}, U_{j-1}^{m+1}, U_{j+1}^m, U_{j-1}^m, U_j^m\right\}$$

Well, we know that $\theta\nu \geq 0$ and that $(1-\theta)\nu \geq 0$. Provided that $1 - 2(1-\theta)\nu \geq 0$, therefore, a point we will return to at the end, we can write that:

$$(1+2\theta\nu)U_j^{m+1} \leq 2\theta\nu U^\star + 2(1-\theta)\nu U^\star + [1 - 2(1-\theta)]U^\star = (1+2\theta\nu)U^\star$$

So:

$$U_j^{m+1} \leq U^\star$$

But since $U^\star$ is a value in the range given above, and U_j^{m+1} was, by assumption, the greatest value in that range, we also have:

$$U^\star \leq U_j^{m+1}$$

This implies that:

$$U_j^{m+1} = U^\star$$

This leads us to conclude that the maximum value is attained at *all* the points over which the maximum for $U^\star$ was taken, since if any of them were less than $U^\star$ we would have $U_j^{m+1} < U^\star$, which would be a contradiction. But now all the surrounding points are equal to U_j^{m+1}, and we can apply the same argument recursively to them until we eventually reach the boundary. So the maximum is attained at a point on the boundary of the range and is therefore equal to $U^n_{\max}$ as required. As noted before, an entirely similar argument will show the same thing for the minimum value.

Of course, all this was dependent on our original proviso, that $1 - 2(1-\theta)\nu \geq 0$, or, to put it another way, that $2\nu(1-\theta) \leq 1$. If and only if this is satisfied, therefore, the θ-scheme obeys the maximum principle given.

(Note that this was almost identical to the proof in the notes, except that here we were dealing with particular ranges of the grid rather than the whole thing at once.)

3. Consider the heat equation $u_t = u_{xx} + u_{yy} + u$ on the unit square $\Omega = (0,1)^2$, and $t \in (0,T]$, subject to homogeneous Dirichlet boundary conditions and the initial condition $u(x,y,0) = u_0(x,y)$.

   Set up an ADI scheme, based on the Crank-Nicolson method, for the numerical solution of this initial boundary value problem, on a uniform spatial mesh of sizes $\Delta x = 1/I$ and $\Delta y = 1/J$, respectively, $I, J \geq 2$.

   Use Fourier analysis to show that your ADI scheme is unconditionally von Neumann stable.

Answer

First let's set up the scheme described. We start by working out what the θ-scheme would be for this problem. We have:

$$\frac{U_{ij}^{m+1} - U_{ij}^{m}}{\Delta t} = (1-\theta)\left(\frac{\delta_x^2 U_{ij}^m}{(\Delta x)^2} + \frac{\delta_y^2 U_{ij}^m}{(\Delta y)^2}\right) + \theta\left(\frac{\delta_x^2 U_{ij}^{m+1}}{(\Delta x)^2} + \frac{\delta_y^2 U_{ij}^{m+1}}{(\Delta y)^2}\right) + U_{ij}^m$$

Rewriting this using $\mu_x = \Delta t/(\Delta x)^2$ and $\mu_y = \Delta t/(\Delta y)^2$, we have:

$$U_{ij}^{m+1} - U_{ij}^m = (1-\theta)(\mu_x \delta_x^2 U_{ij}^m + \mu_y \delta_y^2 U_{ij}^m) + \theta(\mu_x \delta_x^2 U_{ij}^{m+1} + \mu_y \delta_y^2 U_{ij}^{m+1}) + \Delta t U_{ij}^m$$

When $\theta = 1/2$, this becomes the Crank-Nicolson scheme for the problem, and we have:

$$U_{ij}^{m+1}\left(1 - \frac{1}{2}\mu_x\delta_x^2 - \frac{1}{2}\mu_y\delta_y^2\right) = U_{ij}^m\left(1 + \frac{1}{2}\mu_x\delta_x^2 + \frac{1}{2}\mu_y\delta_y^2\right) + U_{ij}^m \Delta t$$

We rewrite this in a way analogous to the notes to get:

$$\left(1 - \frac{1}{2}\mu_x\delta_x^2\right)\left(1 - \frac{1}{2}\mu_y\delta_y^2\right)U_{ij}^{m+1} = \left(1 + \frac{1}{2}\mu_x\delta_x^2\right)\left(\left[1 + \frac{1}{2}\mu_y\delta_y^2\right] + \left[1 + \frac{1}{2}\mu_x\delta_x^2\right]^{-1}\Delta t\right)U_{ij}^m$$

Upon introducing the intermediate level $U^{m+1/2}$, we can rewrite the last equality as:

$$\left(1 - \frac{1}{2}\mu_x\delta_x^2\right)U_{ij}^{m+1/2} = \left(\left[1 + \frac{1}{2}\mu_y\delta_y^2\right] + \left[1 + \frac{1}{2}\mu_x\delta_x^2\right]^{-1}\Delta t\right)U_{ij}^m$$

$$\left(1 - \frac{1}{2}\mu_y\delta_y^2\right)U_{ij}^{m+1} = \left(1 + \frac{1}{2}\mu_x\delta_x^2\right)U_{ij}^{m+1/2}$$

The equivalence is seen by applying $(1 + \frac{1}{2}\mu_x\delta_x^2)$ to the first equation and $(1 - \frac{1}{2}\mu_x\delta_x^2)$ to the second equation.

***

We have our ADI scheme; it remains to show that it's unconditionally von Neumann stable. We use the Fourier mode method to show this, substituting

$$U_{ij}^m = [\lambda(k_x, k_y)]^m \, e^{i(k_x x_i + k_y y_j)}$$

into the scheme, thereby obtaining (if we write $\lambda \equiv \lambda(k_x, k_y)$ for brevity):

$$\begin{aligned}
& \lambda^{m+1}\left[1-\frac{1}{2}\mu_x\left(e^{ik_x\Delta x}-2+e^{-ik_x\Delta x}\right)-\frac{1}{2}\mu_y\left(e^{ik_y\Delta y}-2+e^{-ik_y\Delta y}\right)\right] \\
= \ & \lambda^{m}\left[1+\Delta t+\frac{1}{2}\mu_x\left(e^{ik_x\Delta x}-2+e^{-ik_x\Delta x}\right)+\frac{1}{2}\mu_y\left(e^{ik_y\Delta y}-2+e^{-ik_y\Delta y}\right)\right]
\end{aligned}$$

So

$$\begin{aligned}
\lambda &= \frac{1+\Delta t+\frac{1}{2}\mu_x\left(e^{ik_x\Delta x}-2+e^{-ik_x\Delta x}\right)+\frac{1}{2}\mu_y\left(e^{ik_y\Delta y}-2+e^{-ik_y\Delta y}\right)}{1-\frac{1}{2}\mu_x\left(e^{ik_x\Delta x}-2+e^{-ik_x\Delta x}\right)-\frac{1}{2}\mu_y\left(e^{ik_y\Delta y}-2+e^{-ik_y\Delta y}\right)} \\
&= \frac{1+\frac{1}{2}\mu_x\left(e^{ik_x\Delta x}-2+e^{-ik_x\Delta x}\right)+\frac{1}{2}\mu_y\left(e^{ik_y\Delta y}-2+e^{-ik_y\Delta y}\right)}{1-\frac{1}{2}\mu_x\left(e^{ik_x\Delta x}-2+e^{-ik_x\Delta x}\right)-\frac{1}{2}\mu_y\left(e^{ik_y\Delta y}-2+e^{-ik_y\Delta y}\right)}+C\Delta t
\end{aligned}$$

where clearly

$$C^{-1}=1-\frac{1}{2}\mu_x\left(e^{ik_x\Delta x}-2+e^{-ik_x\Delta x}\right)-\frac{1}{2}\mu_y\left(e^{ik_y\Delta y}-2+e^{-ik_y\Delta y}\right)$$

It suffices to show, therefore, that:

$$\left|\frac{1+\frac{1}{2}\mu_x\left(e^{ik_x\Delta x}-2+e^{-ik_x\Delta x}\right)+\frac{1}{2}\mu_y\left(e^{ik_y\Delta y}-2+e^{-ik_y\Delta y}\right)}{1-\frac{1}{2}\mu_x\left(e^{ik_x\Delta x}-2+e^{-ik_x\Delta x}\right)-\frac{1}{2}\mu_y\left(e^{ik_y\Delta y}-2+e^{-ik_y\Delta y}\right)}\right|\leq 1$$

This comes from the fact that to show von Neumann stability we need to show that $|\lambda(k_x,k_y)|\leq 1+C\Delta t$ for some $C=C(T)$. We've got our $C\Delta t$ bit; all that remains is showing that the rest is no greater than 1.

We observe that we can rewrite this as:

$$\left|\frac{1+\mu_x(\cos k_x\Delta x-1)+\mu_y(\cos k_y\Delta y-1)}{1-\mu_x(\cos k_x\Delta x-1)-\mu_y(\cos k_y\Delta y-1)}\right|\leq 1$$

Which becomes:

$$|1+\mu_x(\cos k_x\Delta x-1)+\mu_y(\cos k_y\Delta y-1)|\leq|1-\mu_x(\cos k_x\Delta x-1)-\mu_y(\cos k_y\Delta y-1)|$$

Let's write $a\equiv\mu_x(\cos k_x\Delta x-1)$ and $b\equiv\mu_y(\cos k_y\Delta y-1)$ for brevity. Then we have:

$$|1+a+b|^2\leq|1-a-b|^2$$

Expanding this out gives us:

$$1+a^2+b^2+2(ab+a+b)\leq 1+a^2+b^2+2(ab-a-b)$$

Cancelling terms on both sides leaves us with:

$$2(a+b)\leq -2(a+b)$$

This is clearly satisfied if and only if $a+b\leq 0$, i.e. if:

$$\mu_x(\cos k_x\Delta x-1)+\mu_y(\cos k_y\Delta y-1)\leq 0$$

But this is always ≤ 0! Consider the fact that $\forall\zeta\cdot\cos\zeta-1\leq 0$, and that $\mu_x>0$ and $\mu_y>0$. Then clearly both terms in the sum are negative (a positive times a negative gives a negative...) and the sum must itself be negative and hence ≤ 0. So the scheme is unconditionally von Neumann stable, as required.