Numerical Solution of Differential Equations
Problem Sheet 5

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1. Consider the implicit Euler scheme

$$\frac{U_j^{n+1}-U_j^n}{\Delta t}+b\frac{U_{j+1}^{n+1}-U_{j-1}^{n+1}}{2\Delta x}=a\frac{U_{j+1}^{n+1}-2U_j^{n+1}+U_{j-1}^{n+1}}{(\Delta x)^2},\quad 1\le j\le J-1,\quad n\ge 0,$$

$$U_0^{n+1}=0,\quad U_J^{n+1}=0,\quad n\ge 0,$$
$$U_j^0=u_0(x_j),\quad 1\le j\le J-1,$$

for the numerical solution of the initial boundary value problem

$$\frac{\partial u}{\partial t}+b\frac{\partial u}{\partial x}=a\frac{\partial^2 u}{\partial x^2},\quad 0<x<1,\quad t>0,$$
$$u(0,t)=0,\quad u(1,t)=0,\quad t>0,$$
$$u(x,0)=u_0(x),\quad 0<x<1,$$

where $a>0$ and b are fixed real numbers. Show that the scheme is unconditionally stable in the ℓ_2 norm.

Show further that the truncation error $|T_j^n|\le C(\Delta t+(\Delta x)^2)$ for all $n\ge 0$ and $1\le j\le J-1$, where C is a constant independent of Δt and Δx, provided that $\partial^2 u/\partial t^2$, $\partial^3 u/\partial x^3$ and $\partial^4 u/\partial x^4$ exist and are bounded functions of x and t, $(x,t)\in[0,1]\times[0,\infty)$.

Answer

We'll do this using the Fourier mode method from the notes. To recap how this works: we write

$$U_j^m=[\lambda(k)]^m\,e^{ikj\Delta x},$$

and note that

$$U_j^{m+1}=[\lambda(k)]^{m+1}\,e^{ikj\Delta x}=\lambda(k)U_j^m,$$

hence

$$\|U^{m+1}\|_{\ell_2}\le\max_k|\lambda(k)|\|U^m\|_{\ell_2}$$

as required. Well, what is $\lambda(k)$ for this particular scheme? We observe that:

$$\frac{\lambda(k)-1}{\Delta t}+\frac{b\lambda(k)(e^{ik\Delta x}-e^{-ik\Delta x})}{2\Delta x}=\frac{a\lambda(k)(e^{ik\Delta x}-2+e^{-ik\Delta x})}{(\Delta x)^2}$$

Simplifying this, we get:

$$\lambda(k)\left[\frac{1}{\Delta t}+\frac{b(e^{ik\Delta x}-e^{-ik\Delta x})}{2\Delta x}-\frac{a(e^{ik\Delta x}-2+e^{-ik\Delta x})}{(\Delta x)^2}\right]=\frac{1}{\Delta t}$$

$$\lambda(k)\left[1+\frac{b\Delta t(e^{ik\Delta x}-e^{-ik\Delta x})}{2\Delta x}-\frac{a\Delta t(e^{ik\Delta x}-2+e^{-ik\Delta x})}{(\Delta x)^2}\right]=1$$

$$\lambda(k)\left[\frac{2(\Delta x)^2 + b\Delta t\Delta x(e^{ik\Delta x} - e^{-ik\Delta x}) - 2a\Delta t(e^{ik\Delta x} - 2 + e^{-ik\Delta x})}{2(\Delta x)^2}\right] = 1$$

$$\lambda(k)\left[\frac{2(\Delta x)^2 + b\Delta t\Delta x(2i\sin k\Delta x) - 2a\Delta t(2i\sin k\Delta x - 2)}{2(\Delta x)^2}\right] = 1$$

$$\lambda(k)\left[\frac{2(\Delta x)^2 + 4a\Delta t + 2i\Delta t(\sin k\Delta x)(b\Delta x - 2a)}{2(\Delta x)^2}\right] = 1$$

$$\lambda(k)\left[\frac{2(\Delta x)^2 + \Delta t[4a + 2i(\sin k\Delta x)(b\Delta x - 2a)]}{2(\Delta x)^2}\right] = 1$$

$$\lambda(k)\left[1 + \mu(2a + i(\sin k\Delta x)(b\Delta x - 2a))\right] = 1$$

$$\lambda(k) = \frac{1}{1 + \mu(2a + i(\sin k\Delta x)(b\Delta x - 2a))}$$

For unconditional stability, we want that $|\lambda(k)| \leq 1$ for all $k \in [-\pi/\Delta x, \pi/\Delta x]$, so we want:

$$|1 + \mu(2a + i(\sin k\Delta x)(b\Delta x - 2a))| \geq 1$$

In other words:

$$1^2 \leq |(1 + 2a\mu) + i(\sin k\Delta x)(b\Delta x - 2a)|^2 = (1 + 2a\mu)^2 + [(\sin k\Delta x)(b\Delta x - 2a)]^2$$

Well this is clearly satisfied, since the above is greater than or equal to:

$$1 + 4a\mu + 4a^2\mu^2 > 1$$

This last inequality follows directly from the fact that $a > 0$ and $\mu = \Delta t/(\Delta x)^2 > 0$. So the scheme is unconditionally stable, as required.

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We now move on to the bit involving the truncation error. The truncation error for this scheme is defined by

$$T_j^n = \frac{u_j^{n+1} - u_j^n}{\Delta t} + b\frac{u_{j+1}^{n+1} - u_{j-1}^{n+1}}{2\Delta x} - a\frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{(\Delta x)^2},$$

where

$$u_j^n = u(x_j, t_n).$$

We can Taylor expand this to get something we can work with. As per the notes:

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \left[u_t + \frac{1}{24}(\Delta t)^2 u_{ttt} + \dots \right]_j^{n+1/2}$$

We recall that the Taylor series expansion of a real function of two variables is given by:

$$f(x + \delta x, y + \delta y) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{(\delta x)^j}{j!} \frac{(\delta y)^k}{k!} \frac{\partial^j}{\partial x^j} \frac{\partial^k}{\partial y^k} f(x, y)$$

So, expanding around $(x_j, t_{n+1/2})$, we have (since $\delta x = \Delta x$ and $\delta t = \Delta t/2$):

$$\begin{aligned} u_{j+1}^{n+1} = [u \quad &+ \quad \left(u_x \Delta x + u_t \frac{\Delta t}{2} \right) \\ &+ \quad \frac{1}{2!} \left((\Delta x)^2 u_{xx} + (\Delta x \Delta t) u_{xt} + \frac{(\Delta t)^2}{4} u_{tt} \right) \\ &+ \quad \frac{1}{3!} \left((\Delta x)^3 u_{xxx} + \frac{3}{2} (\Delta x)^2 (\Delta t) u_{xxt} + \frac{3}{4} \Delta x (\Delta t)^2 u_{xtt} + \frac{(\Delta t)^3}{8} u_{ttt} \right) \\ &+ \quad \frac{1}{4!} \left((\Delta x)^4 u_{xxxx} + \frac{4}{2} (\Delta x)^3 (\Delta t) u_{xxxt} + \frac{6}{4} (\Delta x)^2 (\Delta t)^2 u_{xxtt} + \right. \\ &\qquad \left. \frac{4}{8} \Delta x (\Delta t)^3 u_{xttt} + \frac{1}{16} (\Delta t)^4 u_{tttt} \right) \\ &+ \quad \dots]_j^{n+1/2} \end{aligned}$$

Analogously (using $\delta x = -\Delta x$ and $\delta t = \Delta t/2$):

$$\begin{aligned} u_{j-1}^{n+1} = [u \quad &+ \quad \left(-u_x \Delta x + u_t \frac{\Delta t}{2} \right) \\ &+ \quad \frac{1}{2!} \left((\Delta x)^2 u_{xx} - (\Delta x \Delta t) u_{xt} + \frac{(\Delta t)^2}{4} u_{tt} \right) \\ &+ \quad \frac{1}{3!} \left(-(\Delta x)^3 u_{xxx} + \frac{3}{2} (\Delta x)^2 (\Delta t) u_{xxt} - \frac{3}{4} \Delta x (\Delta t)^2 u_{xtt} + \frac{(\Delta t)^3}{8} u_{ttt} \right) \\ &+ \quad \frac{1}{4!} \left((\Delta x)^4 u_{xxxx} - 2 (\Delta x)^3 (\Delta t) u_{xxxt} + \frac{3}{2} (\Delta x)^2 (\Delta t)^2 u_{xxtt} - \right. \\ &\qquad \left. \frac{1}{2} \Delta x (\Delta t)^3 u_{xttt} + \frac{1}{16} (\Delta t)^4 u_{tttt} \right) \\ &+ \quad \dots]_j^{n+1/2} \end{aligned}$$

Furthermore (using $\delta x = 0$ and $\delta t = \Delta t/2$):

$$u_j^{n+1} = \left[u + u_t \frac{\Delta t}{2} + \frac{1}{2!} u_{tt} \frac{(\Delta t)^2}{4} + \frac{1}{3!} u_{ttt} \frac{(\Delta t)^3}{8} + \frac{1}{4!} u_{tttt} \frac{(\Delta t)^4}{16} + \dots \right]_j^{n+1/2}$$

So far, so good. Now:

$$\begin{aligned}\frac{u_{j+1}^{n+1}-u_{j-1}^{n+1}}{2\Delta x} \approx \frac{1}{\Delta x} \quad [& \quad u_x \Delta x \\ + & \quad \frac{1}{2!}((\Delta x \Delta t) u_{xt}) \\ + & \quad \frac{1}{3!}((\Delta x)^3 u_{xxx} + \frac{3}{4}\Delta x (\Delta t)^2 u_{xtt}) \\ + & \quad \frac{1}{4!}(2(\Delta x)^3 (\Delta t) u_{xxxt} + \frac{1}{2}\Delta x (\Delta t)^3 u_{xttt}) \\ + & \quad \ldots]_j^{n+1/2}\end{aligned}$$

Thus:

$$\begin{aligned}\frac{u_{j+1}^{n+1}-u_{j-1}^{n+1}}{2\Delta x} \approx [u_x & + \quad \frac{1}{2}\Delta t u_{xt} \\ & + \quad \frac{1}{6}((\Delta x)^2 u_{xxx} + \frac{3}{4}(\Delta t)^2 u_{xtt}) \\ & + \quad \frac{1}{24}(2(\Delta x)^2 (\Delta t) u_{xxxt} + \frac{1}{2}(\Delta t)^3 u_{xttt}) \\ & + \quad \ldots]_j^{n+1/2}\end{aligned}$$

Now for the really fun bit.[1] We calculate that:

$$\begin{aligned}\frac{u_{j+1}^{n+1}-2u_j^{n+1}+u_{j-1}^{n+1}}{(\Delta x)^2} \approx \frac{1}{(\Delta x)^2}[2(u & + \quad u_t \frac{\Delta t}{2} \\ & + \quad \frac{1}{2}((\Delta x)^2 u_{xx} + \frac{(\Delta t)^2}{4} u_{tt}) \\ & + \quad \frac{1}{6}(\frac{3}{2}(\Delta x)^2 (\Delta t) u_{xxt} + \frac{(\Delta t)^3}{8} u_{ttt}) \\ & + \quad \frac{1}{24}((\Delta x)^4 u_{xxxx} + \frac{3}{2}(\Delta x)^2 (\Delta t)^2 u_{xxtt} + \frac{1}{16}(\Delta t)^4 u_{tttt}) \\ & + \quad \ldots) \\ -2(u & + \quad u_t \frac{\Delta t}{2} + \frac{1}{2} u_{tt} \frac{(\Delta t)^2}{4} \\ & + \quad \frac{1}{6} u_{ttt} \frac{(\Delta t)^3}{8} + \frac{1}{24} u_{tttt} \frac{(\Delta t)^4}{16} + \ldots)]_j^{n+1/2}\end{aligned}$$

This looks fairly gory, but most things cancel quite nicely to give us:

$$\frac{u_{j+1}^{n+1}-2u_j^{n+1}+u_{j-1}^{n+1}}{(\Delta x)^2} \approx \left[u_{xx} + \frac{1}{2}(\Delta t) u_{xxt} + \frac{1}{12}(\Delta x)^2 u_{xxxx} + \frac{1}{8}(\Delta t)^2 u_{xxtt} + \ldots\right]_j^{n+1/2}$$

[1] Sometimes referred to in the literature as 'the bit that makes grown men (and typesetters) weep'...but I digress. Incidentally, it's ambiguous as to whether that means grown men and grown typesetters, or just grown men and (normal-sized) typesetters. Readers are free to interpret this however they wish. I strongly suspect that it doesn't matter all that much.

We're on the home leg at last! We calculate that:

$$
\begin{aligned}
T_j^n \quad = & \quad [u_t + \frac{1}{24}(\Delta t)^2 u_{ttt} + \ldots]_j^{n+1/2} \\
+ & \quad b[u_x + \frac{1}{2}\Delta t u_{xt} \\
& \quad + \frac{1}{6}((\Delta x)^2 u_{xxx} + \frac{3}{4}(\Delta t)^2 u_{xtt}) \\
& \quad + \frac{1}{24}(2(\Delta x)^2(\Delta t) u_{xxxt} + \frac{1}{2}(\Delta t)^3 u_{xttt}) \\
& \quad + \ \ldots]_j^{n+1/2} \\
- & \quad a[u_{xx} + \frac{1}{2}(\Delta t) u_{xxt} + \frac{1}{12}(\Delta x)^2 u_{xxxx} + \frac{1}{8}(\Delta t)^2 u_{xxtt} + \ldots]_j^{n+1/2}
\end{aligned}
$$

We observe that $u_t + bu_x - au_{xx} = 0$ since $u_t + bu_x = au_{xx}$, so those get cancelled and we end up with an expression for T_j^n whose highest-order terms are in Δt and $(\Delta x)^2$, as required. So the truncation error is $O(\Delta t + (\Delta x)^2)$ as required, which is in this instance an equivalent statement to saying that $\exists C \cdot |T_j^n| \leq C(\Delta t + (\Delta x)^2)$ for all $n \geq 0$ and $1 \leq j \leq J - 1$. (Note that the only change from our usual definition of big-O notation is that here $n \geq 0$ rather than $n \geq n_0$; in other words, $n_0 = 0$ in this case.)

2. Consider the system of linear equations

$$-a_jU_{j-1} + b_jU_j - c_jU_{j+1} = d_j, \quad j = 1, \ldots, J-1,$$

with

$$U_0 = 0, \quad U_J = 0,$$

where $a_j > 0$, $b_j > 0$, $c_j > 0$ and $b_j > a_j + c_j$ for all j.

(a) Show that

$$U_j = e_jU_{j+1} + f_j, \quad j = J-1, J-2, \ldots, 1, \tag{1}$$

where

$$e_j = \frac{c_j}{b_j - a_je_{j-1}}, \quad f_j = \frac{d_j + a_jf_{j-1}}{b_j - a_je_{j-1}}, \quad j = 1, 2, \ldots, J-1, \tag{2}$$

with $e_0 = 0$ and $f_0 = 0$. This method for the solution of the linear system of equations $(1), (2)$ is called the *Thomas algorithm*.

(b) Show by induction that $0 < e_j < 1$ for $j = 1, 2, \ldots, J-1$. Show further that the conditions

$$b_j > 0, \quad b_j \geq |a_j| + |c_j|, \quad j = 1, 2, \ldots, J-1,$$

are sufficient to ensure that $|e_j| \leq 1$ for $j = 1, 2, \ldots, J-1$. What do you think the practical significance of the last inequality is regarding the sensitivity of the algorithm to rounding errors?

Answer

(a) We first write the system of equations in matrix form as:

$$\begin{pmatrix} b_1 & -c_1 & 0 & 0 & \ldots & 0 & 0 & 0 \\ -a_2 & b_2 & -c_2 & 0 & \ldots & 0 & 0 & 0 \\ 0 & -a_3 & b_3 & -c_3 & \ldots & 0 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & \ldots & b_{J-3} & -c_{J-3} & 0 \\ 0 & 0 & 0 & 0 & \ldots & -a_{J-2} & b_{J-2} & -c_{J-2} \\ 0 & 0 & 0 & 0 & \ldots & 0 & -a_{J-1} & b_{J-1} \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \\ U_3 \\ \ldots \\ U_{J-3} \\ U_{J-2} \\ U_{J-1} \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ \ldots \\ d_{J-3} \\ d_{J-2} \\ d_{J-1} \end{pmatrix}$$

Now, we use an inductive argument as per the notes to show our result. First the base case. When $j = 1$, the equation is:

$$b_1U_1 - c_1U_2 = d_1$$

Rearranging this gives us:

$$U_1 = \frac{c_1}{b_1}U_2 + \frac{d_1}{b_1} = \underbrace{\left(\frac{c_1}{b_1 - a_1e_0}\right)}_{e_1} U_2 + \underbrace{\left(\frac{d_1 + a_1f_0}{b_1 - a_1e_0}\right)}_{f_1}$$

So the base case is certainly satisfied. Now, for the inductive step, suppose that equation j has been reduced to $U_j - e_j U_{j+1} = f_j$ (i.e. the result holds for equation j) and that the next equation is still in its original form, namely:

$$-a_{j+1}U_j + b_{j+1}U_{j+1} - c_{j+1}U_{j+2} = d_{j+1}$$

We want to eliminate U_j from this to get something in terms of U_{j+1} and U_{j+2} only. To do this, we add a_{j+1} times the above row to this one, obtaining:

$$(b_{j+1} - a_{j+1}e_j)U_{j+1} - c_{j+1}U_{j+2} = d_{j+1} + a_{j+1}f_j$$

Rearranging this gives us:

$$U_{j+1} = \underbrace{\left(\frac{c_{j+1}}{b_{j+1} - a_{j+1}e_j}\right)}_{e_{j+1}} U_{j+2} + \underbrace{\left(\frac{d_{j+1} + a_{j+1}f_j}{b_{j+1} - a_{j+1}e_j}\right)}_{f_{j+1}}$$

This completes the inductive proof, so the result holds for all relevant j.

(b) For the base case, we observe that:

$$e_1 = \frac{c_1}{b_1 - a_1 \cdot 0} = \frac{c_1}{b_1}$$

This is clearly > 0, since both b_1 and c_1 are. Moreover, since for all j, $b_j > a_j + c_j > c_j$, clearly $b_1 > c_1$ and hence $c_1/b_1 < 1$.

Now for the inductive step. Assuming that $0 < e_{j-1} < 1$, we need to show that $0 < e_j < 1$. Well, we note that

$$e_j = \frac{c_j}{b_j - a_j e_{j-1}},$$

so since $e_{j-1} \in (0, 1)$,

$$\frac{c_j}{b_j} \leq e_j \leq \frac{c_j}{b_j - a_j}.$$

Now clearly $c_j/b_j > 0$, since both b_j and c_j are, so $e_j > 0$. Furthermore, $c_j/(b_j - a_j) < 1$, since $b_j > a_j + c_j$, so $e_j < 1$. Whence $0 < e_j < 1$, as required, and our result holds by induction.

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We now do a similar proof with the new conditions. The base case isn't too tricky. Noting that

$$|e_1| = \left|\frac{c_1}{b_1}\right| = \frac{|c_1|}{|b_1|},$$

we observe that this is clearly ≤ 1 since $|b_1| \geq |(|a_1| + |c_1|)| = |a_1| + |c_1| \geq |c_1|$.

Now for the inductive step. Assuming that $|e_{j-1}| \leq 1$, we must show that $|e_j| \leq 1$. So:

$$|e_j| = \left|\frac{c_j}{b_j - a_j e_{j-1}}\right| = \frac{|c_j|}{|b_j - a_j e_{j-1}|}$$

Now, using the triangle inequality, namely that $\forall x, y \cdot |x+y| \leq |x| + |y|$, and the usual identities $|-x| = |x|$ and $|xy| = |x||y|$, we rewrite the denominator, giving us:

$$|e_j| = \frac{|c_j|}{|b_j| + |-a_j e_{j-1}|} = \frac{|c_j|}{|b_j| + |a_j e_{j-1}|} = \frac{|c_j|}{|b_j| + |a_j||e_{j-1}|}$$

We know that $|e_{j-1}| \in [0, 1]$, so we can deduce that:

$$\frac{|c_j|}{|b_j| + |a_j|} \leq |e_j| \leq \frac{|c_j|}{|b_j|}$$

Now, analogously to the base case, we observe that $|b_j| \geq |(|a_j|+|c_j|)| = |a_j|+|c_j| \geq |c_j|$, so $|c_j|/|b_j| \leq 1$ and hence $|e_j| \leq 1$ and our result holds for all relevant j. (It's worth noting, for completeness, that the left-hand inequality doesn't tell us anything terribly interesting here.)

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TODO: I'm not sure about the practical significance bit.

3. Consider the θ-method for the numerical solution of the initial value problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 1, \quad t > 0,$$

$$u(x,0) = u_0(x), \quad 0 < x < 1,$$

subject to homogeneous Dirichlet boundary conditions at $x = 0$ and $x = 1$. Suppose that the parameter θ has been chosen according to the formula

$$\theta = \frac{1}{2} + \frac{(\Delta x)^2}{12\Delta t}.$$

Show that the resulting scheme is unconditionally stable in the ℓ_2 norm and has a truncation error which is $O((\Delta t)^2 + (\Delta x)^2)$, provided that derivatives of u of sufficiently high order exist and are bounded functions of x and t, $(x,t) \in [0,1] \times [0,\infty)$.

Answer

We'll leave the unconditional stability bit till later and concentrate on the truncation error. As per the notes,[2] the truncation error for the θ-method applied to a heat equation problem is given by:

$$\begin{aligned} T_j^m &= \left[\left(\frac{1}{2} - \theta\right)\Delta t u_{xxt} - \frac{1}{12}(\Delta x)^2 u_{xxxx}\right]_j^{m+1/2} \\ &+ \left[\frac{1}{24}(\Delta t)^2 u_{ttt} - \frac{1}{8}(\Delta t)^2 u_{xxtt}\right]_j^{m+1/2} \\ &+ \left[\frac{1}{12}\left(\frac{1}{2} - \theta\right)\Delta t(\Delta x)^2 u_{xxxxt} - \frac{2}{6!}(\Delta x)^4 u_{xxxxxx}\right]_j^{m+1/2} \\ &+ \ldots \end{aligned}$$

(The original Taylor expansion had a $[u_t - u_{xx}]_j^{m+1/2}$ term in it, but that equalled 0 since the equation to which we're applying the method is $u_t = u_{xx}$.) Now, plugging the value we've been given for θ into the above equation for the truncation error, we get:

$$\begin{aligned} T_j^m &= \left[\left(-\frac{(\Delta x)^2}{12\Delta t}\right)\Delta t u_{xxt} - \frac{1}{12}(\Delta x)^2 u_{xxxx}\right]_j^{m+1/2} \\ &+ \left[\frac{1}{24}(\Delta t)^2 u_{ttt} - \frac{1}{8}(\Delta t)^2 u_{xxtt}\right]_j^{m+1/2} \\ &+ \ldots \end{aligned}$$

[2](And thus avoiding redoing the Taylor expansion, which was nasty enough the first time...)

So:

$$\begin{aligned} T_j^m &= \left[-\frac{1}{12}(\Delta x)^2(u_{xxt} + u_{xxxx})\right]_j^{m+1/2} \\ &+ \left[\frac{1}{24}(\Delta t)^2 u_{ttt} - \frac{1}{8}(\Delta t)^2 u_{xxtt}\right]_j^{m+1/2} \\ &+ \dots \end{aligned}$$

This implies that the truncation error is $O((\Delta t)^2 + (\Delta x)^2)$, as required.

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Now for the unconditional stability bit. First we write the scheme out and plug our value of θ in so we know what we're dealing with:

$$\begin{aligned} \frac{U_j^{m+1} - U_j^m}{\Delta t} &= (1-\theta)\frac{U_{j+1}^m - 2U_j^m + U_{j-1}^m}{(\Delta x)^2} \\ &+ \theta\frac{U_{j+1}^{m+1} - 2U_j^{m+1} + U_{j-1}^{m+1}}{(\Delta x)^2} \\ &= \left(\frac{1}{2} - \frac{(\Delta x)^2}{12\Delta t}\right)\frac{U_{j+1}^m - 2U_j^m + U_{j-1}^m}{(\Delta x)^2} \\ &+ \left(\frac{1}{2} + \frac{(\Delta x)^2}{12\Delta t}\right)\frac{U_{j+1}^{m+1} - 2U_j^{m+1} + U_{j-1}^{m+1}}{(\Delta x)^2} \end{aligned}$$

So, letting $\mu = \Delta t/(\Delta x)^2$:

$$\begin{aligned} U_j^{m+1} - U_j^m = &\left(\frac{1}{2} - \frac{1}{12\mu}\right)\mu(U_{j+1}^m - 2U_j^m + U_{j-1}^m) \\ + &\left(\frac{1}{2} + \frac{1}{12\mu}\right)\mu(U_{j+1}^{m+1} - 2U_j^{m+1} + U_{j-1}^{m+1}) \end{aligned}$$

Now we use the same Fourier mode argument that we did in the first question. Writing

$$U_j^m = [\lambda(k)]^m e^{ikj\Delta x},$$

we note that

$$\begin{aligned} \lambda(k) - 1 = &\left(\frac{1}{2} - \frac{1}{12\mu}\right)\mu(e^{ik\Delta x} - 2 + e^{-ik\Delta x}) \\ + &\left(\frac{1}{2} + \frac{1}{12\mu}\right)\mu\lambda(k)(e^{ik\Delta x} - 2 + e^{-ik\Delta x}). \end{aligned}$$

Let's simplify this:

$$\begin{aligned} 12\lambda(k) - 12 = & \quad (6\mu - 1)(e^{ik\Delta x} - 2 + e^{-ik\Delta x}) \\ + & \quad (6\mu + 1)\lambda(k)(e^{ik\Delta x} - 2 + e^{-ik\Delta x}) \\ = & \quad [(6\mu - 1) + (6\mu + 1)\lambda(k)](2i \sin k\Delta x - 2) \end{aligned}$$

So:

$$\begin{aligned} 6\lambda(k) - 6 &= [(6\mu - 1) + (6\mu + 1)\lambda(k)](i \sin k\Delta x - 1) \\ &= (6\mu - 1)(i \sin k\Delta x - 1) + (6\mu + 1)(i \sin k\Delta x - 1)\lambda(k) \end{aligned}$$

And hence:

$$\lambda(k) = \frac{6 + (6\mu - 1)(i \sin k\Delta x - 1)}{6 - (6\mu + 1)(i \sin k\Delta x - 1)}$$

For unconditional stability, we require that for all $k \in [-\pi/\Delta x, \pi/\Delta x]$:

$$|\lambda(k)| \leq 1$$

We calculate as follows:

$$\begin{aligned} |\lambda(k)|^2 &= \frac{|6 + (6\mu - 1)(i \sin k\Delta x - 1)|^2}{|6 - (6\mu + 1)(i \sin k\Delta x - 1)|^2} \\ &= \frac{|[6 - (6\mu - 1)] + i[(6\mu - 1) \sin k\Delta x]|^2}{|[6 + (6\mu + 1)] + i[-(6\mu + 1) \sin k\Delta x]|^2} \\ &= \frac{[6 - (6\mu - 1)]^2 + [(6\mu - 1) \sin k\Delta x]^2}{[6 + (6\mu + 1)]^2 + [-(6\mu + 1) \sin k\Delta x]^2} \\ &= \frac{36 - 12(6\mu - 1) + (6\mu - 1)^2 + (6\mu - 1)^2 \sin^2 k\Delta x}{36 + 12(6\mu + 1) + (6\mu + 1)^2 + (6\mu + 1)^2 \sin^2 k\Delta x} \\ &= \frac{36 - 12(6\mu - 1) + (6\mu - 1)^2(1 + \sin^2 k\Delta x)}{36 + 12(6\mu + 1) + (6\mu + 1)^2(1 + \sin^2 k\Delta x)} \end{aligned}$$

For unconditional stability, we need to show that:

$$36 - 12(6\mu - 1) + (6\mu - 1)^2(1 + \sin^2 k\Delta x) \leq 36 + 12(6\mu + 1) + (6\mu + 1)^2(1 + \sin^2 k\Delta x)$$

This reduces to:

$$-12(6\mu - 1) + (6\mu - 1)^2(1 + \sin^2 k\Delta x) \leq 12(6\mu + 1) + (6\mu + 1)^2(1 + \sin^2 k\Delta x)$$

Or alternatively:

$$(1 + \sin^2 k\Delta x)[(6\mu + 1)^2 - (6\mu - 1)^2] + 12[(6\mu + 1) + (6\mu - 1)] \geq 0$$

Well, since $\mu > 0$, we note that $6\mu + 1 > 1$ and $6\mu - 1 > -1$. Thus, since also $6\mu + 1 > 6\mu - 1$, we note that $(6\mu + 1)^2 > (6\mu - 1)^2$, whence the first part above is obviously > 0. It suffices to show, therefore, that:

$$(6\mu + 1) + (6\mu - 1) \geq 0 \Leftrightarrow 12\mu \geq 0$$

This is obviously satisfied since $\mu > 0$. So the method is unconditionally stable in the ℓ_2 norm, as required.