Numerical Solution of Differential Equations
Problem Sheet 4

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Suppose that we have discrete data $\{U_j\}$ defined on an infinite grid $x_j = j\Delta x$, $j = 0, \pm 1, \pm 2, \ldots$. Let δ and μ be the discrete differentiation and smoothing operators defined by

$$(\delta U)_j = (U_{j+1} - U_{j-1})/(2\Delta x), \quad (\mu U)_j = (U_{j+1} + U_{j-1})/2.$$

1. Determine the functions δU, δV, μU, μV for $U = (\ldots, 1, -1, 1, -1, 1, -1, 1, \ldots)$ and $V = (\ldots, 1, 0, -1, 0, 1, 0, -1, 0, \ldots)$.

   Answer

   We note that $\forall j \cdot U_{j+1} = U_{j-1}$, since either $U_{j+1} = U_{j-1} = 1$ or $U_{j+1} = U_{j-1} = -1$. Thus $(\delta U)_j \equiv 0$. Furthermore, $(\mu U)_j = (U_{j+1} + U_{j-1})/2 = 2U_{j+1}/2 = U_{j+1}$ $(= U_{j-1})$.

   We further note that $\forall j \cdot V_{j-1} = -V_{j+1}$. Thus $(\delta V)_j = (V_{j+1} - V_{j-1})/(2\Delta x) = 2V_{j+1}/(2\Delta x) = V_{j+1}/\Delta x$. Furthermore, $(\mu V)_j = (V_{j+1} + V_{j-1})/2 \equiv 0$.

2. Determine what effect δ and μ have on the function U defined by $U_j = e^{ikx_j}$, $j = 0, \pm 1, \pm 2, \ldots$, where k is a real constant (the wave number).

   Answer

   We calculate as follows:

$$\begin{aligned}
(\delta U)_j &= (U_{j+1} - U_{j-1})/(2\Delta x) \\
&= \left(e^{ikx_{j+1}} - e^{ikx_{j-1}}\right)/(2\Delta x) \\
&= \left(e^{ik(x_j+\Delta x)} - e^{ik(x_j-\Delta x)}\right)/(2\Delta x) \\
&= \left(e^{ikx_j}(e^{ik\Delta x} - e^{-ik\Delta x})\right)/(2\Delta x) \\
&= U_j \left(\frac{e^{ik\Delta x} - e^{-ik\Delta x}}{2\Delta x}\right) \\
&= U_j \left(\frac{\sinh ik\Delta x}{\Delta x}\right) \\
&= U_j \left(\frac{i \sin k\Delta x}{\Delta x}\right)
\end{aligned}$$

We now calculate the effect of μ:

$$\begin{aligned}
(\mu U)_j &= (U_{j+1} + U_{j-1})/2 \\
&= (e^{ikx_{j+1}} + e^{ikx_{j-1}})/2 \\
&= (e^{ik(x_j+\Delta x)} + e^{ik(x_j-\Delta x)})/2 \\
&= (e^{ikx_j}(e^{ik\Delta x} + e^{-ik\Delta x}))/2 \\
&= U_j \left(\frac{e^{ik\Delta x} + e^{-ik\Delta x}}{2} \right) \\
&= U_j \cosh ik\Delta x \\
&= U_j \cos k\Delta x
\end{aligned}$$

3. The semidiscrete Fourier transform of a function U defined on the infinite grid $x_j = j\Delta x$, $j = 0, \pm 1, \pm 2, \ldots$, is the function $k \mapsto \hat{U}(k)$, $k \in [-\pi/\Delta x, \pi/\Delta x]$, defined by

$$\hat{U}(k) = \Delta x \sum_{j=-\infty}^{\infty} e^{-ikx_j} U_j.$$

[The reason for the restriction on k is that the wave numbers $|k| > \pi/\Delta x$ are not resolvable on a grid of spacing Δx; this is the phenomenon of *aliasing*.]

Show that the inverse of the semidiscrete Fourier transform is given by the formula

$$U_j = \frac{1}{2\pi} \int_{-\pi/\Delta x}^{\pi/\Delta x} e^{ikj\Delta x} \hat{U}(k)\, dk.$$

Describe the relationship between $\hat{U}(k)$, and $\widehat{\delta U}(k)$ and $\widehat{\mu U}(k)$. [Note that this is a restatement of Question 2.]

The ratios $\widehat{\delta U}/\hat{U}$ and $\widehat{\mu U}/\hat{U}$ are referred to as *Fourier multipliers*. Sketch the graphs of these Fourier multipliers as functions of $k \in [-\pi/\Delta x, \pi/\Delta x]$.

One would think that applying μ repeatedly to U should lead to a function that is much smoother than U. Explain this effect by considering a sketch of the multiplier function $\widehat{\mu^m U}/\hat{U}$ for $m \gg 1$. Your analysis should reveal that taking successive powers of μ is not a perfect smoothing procedure. Explain.

Answer

The most obvious way to go about showing that this is the inverse is just to plug one equation into the other and try and work the whole thing through. Accordingly, we plug the definition of $\hat{U}(k)$ into the equation giving U_j and show that it does indeed work out to give U_j as the result:

$$
\begin{aligned}
&\frac{1}{2\pi}\int_{-\pi/\Delta x}^{\pi/\Delta x} e^{ikj\Delta x} \underbrace{\left[\Delta x \sum_{m=-\infty}^{\infty} e^{-ikx_m} U_m\right]}_{\hat{U}(k)} dk \\
=\;& \frac{\Delta x}{2\pi}\int_{-\pi/\Delta x}^{\pi/\Delta x} \left[\sum_{m=-\infty}^{\infty} e^{ikj\Delta x} e^{-ikm\Delta x} U_m\right] dk \\
=\;& \frac{\Delta x}{2\pi}\int_{-\pi/\Delta x}^{\pi/\Delta x} \left[\sum_{m=-\infty}^{\infty} e^{ik\Delta x(j-m)} U_m\right] dk \\
=\;& \frac{\Delta x}{2\pi}\sum_{m=-\infty}^{\infty} \left[U_m \int_{-\pi/\Delta x}^{\pi/\Delta x} e^{ik\Delta x(j-m)}\, dk\right] \\
=\;& \frac{\Delta x}{2\pi}\sum_{m=-\infty}^{\infty} \left(\frac{U_m}{i\Delta x(j-m)} \left[e^{ik\Delta x(j-m)}\right]_{-\pi/\Delta x}^{\pi/\Delta x}\right) \\
=\;& \frac{1}{2\pi i}\sum_{m=-\infty}^{\infty} \left(\frac{U_m}{j-m} \left[e^{ik\Delta x(j-m)}\right]_{-\pi/\Delta x}^{\pi/\Delta x}\right) \\
=\;& \frac{1}{2\pi i}\sum_{m=-\infty}^{\infty} \left(\frac{U_m}{j-m} \left[e^{i(\pi/\Delta x)\Delta x(j-m)} - e^{i(-\pi/\Delta x)\Delta x(j-m)}\right]\right) \\
=\;& \frac{1}{2\pi i}\sum_{m=-\infty}^{\infty} \left(\frac{U_m}{j-m} \left[e^{i\pi(j-m)} - e^{-i\pi(j-m)}\right]\right) \\
=\;& \sum_{m=-\infty}^{\infty} \left(\frac{U_m}{\pi(j-m)} \left[\frac{e^{i\pi(j-m)} - e^{-i\pi(j-m)}}{2i}\right]\right) \\
=\;& \sum_{m=-\infty}^{\infty} \left(U_m \left[\frac{\sin \pi(j-m)}{\pi(j-m)}\right]\right)
\end{aligned}
$$

This is looking vaguely promising. We note by Taylor expansion[1] that $(\sin x)/x$ is:

$$1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \ldots$$

In particular, this implies that when $m = j$, i.e. when $\pi(j-m) = 0$, the coefficient of U_j is 1 (as we'd expect). It remains to show that the other coefficients are zero. Well, suppose $m \neq j$, then $(\sin \pi(j-m))/(\pi(j-m)) = 0/(\pi(j-m)) = 0$, since $j - m$ is integral (and hence $\pi(j-m)$ is some multiple of π, whose sine is certainly 0) and $j - m \neq 0$ so there's no problem dividing by it. In conclusion, therefore, our final equation above is equal to U_j and this implies that the formula we were given for the inverse was the right one.

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Now let's look at $\widehat{\delta U}(k)$ and $\widehat{\mu U}(k)$. The first of these is given by:

$$\begin{aligned}
&\widehat{\delta U}(k) \\
= \quad & \Delta x \sum_{j=-\infty}^{\infty} e^{-ikx_j} (\delta U)_j \\
= \quad & \Delta x \sum_{j=-\infty}^{\infty} e^{-ikx_j} (U_{j+1} - U_{j-1})/(2\Delta x) \\
= \quad & \frac{1}{2}\left(\left[\sum_{j=-\infty}^{\infty} e^{-ikj\Delta x} U_{j+1}\right] - \left[\sum_{j=-\infty}^{\infty} e^{-ikj\Delta x} U_{j-1}\right]\right) \\
= \quad & \frac{1}{2}\left(e^{ik\Delta x}\left[\sum_{j=-\infty}^{\infty} e^{-ik(j+1)\Delta x} U_{j+1}\right] - e^{-ik\Delta x}\left[\sum_{j=-\infty}^{\infty} e^{-ik(j-1)\Delta x} U_{j-1}\right]\right) \\
= \quad & \frac{1}{2}\left(e^{ik\Delta x}\left[\sum_{j=-\infty}^{\infty} e^{-ikj\Delta x} U_{j}\right] - e^{-ik\Delta x}\left[\sum_{j=-\infty}^{\infty} e^{-ikj\Delta x} U_{j}\right]\right) \\
= \quad & \left(\frac{e^{ik\Delta x} - e^{-ik\Delta x}}{2}\right) \sum_{j=-\infty}^{\infty} e^{-ikj\Delta x} U_j \\
= \quad & \left(\frac{e^{ik\Delta x} - e^{-ik\Delta x}}{2\Delta x}\right) \Delta x \sum_{j=-\infty}^{\infty} e^{-ikj\Delta x} U_j \\
= \quad & \left(\frac{i \sin k\Delta x}{\Delta x}\right) \hat{U}(k)
\end{aligned}$$

A similar (and thus probably repetitive) argument will doubtless show that:

$$\widehat{\mu U}(k) = \hat{U}(k) \cos k\Delta x$$

[1]Well, by looking at the Taylor expansion of $\sin x$ in my P6 book and dividing through by x, at any rate...

The relationships between $\hat{U}(k)$, $\widehat{\delta U}(k)$ and $\widehat{\mu U}(k)$ are thus the same as the relationships in the previous question, as suggested.

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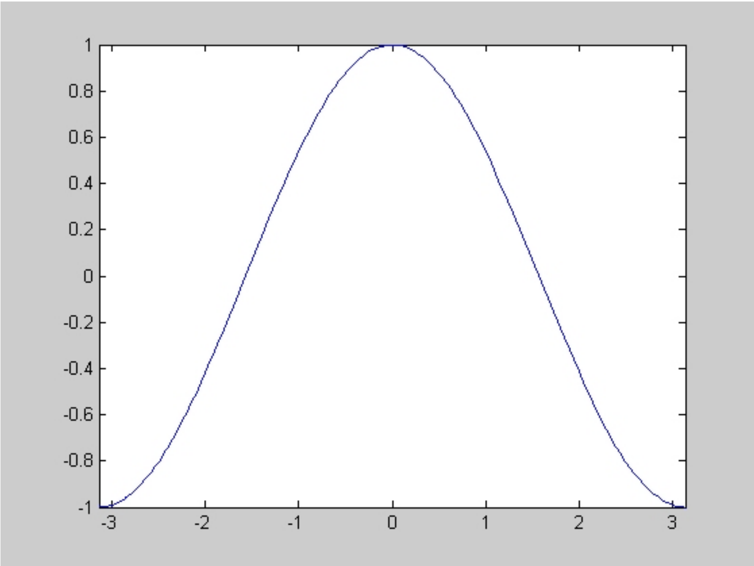
We note that

$$\widehat{\delta U}/\hat{U} = \frac{i \sin k\Delta x}{\Delta x}$$

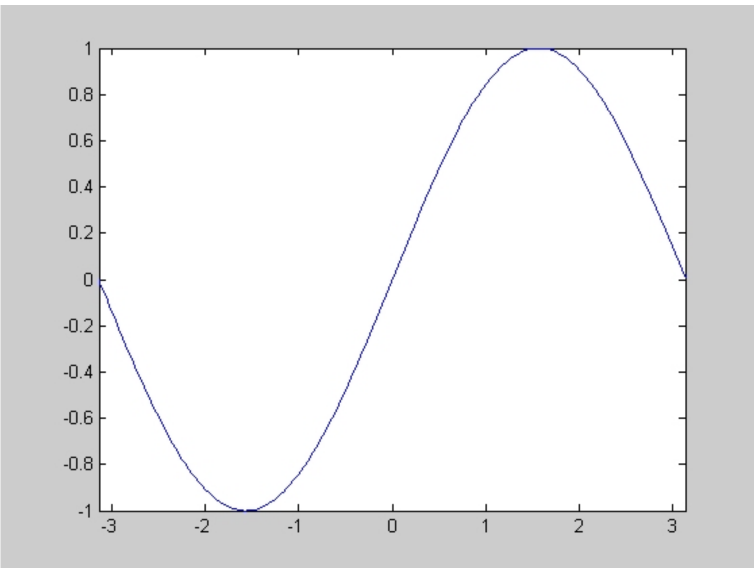
and that

$$\widehat{\mu U}/\hat{U} = \cos k\Delta x.$$

Sketching the second simply gives us a cosine curve in the range $[-\pi, \pi]$, which looks like:



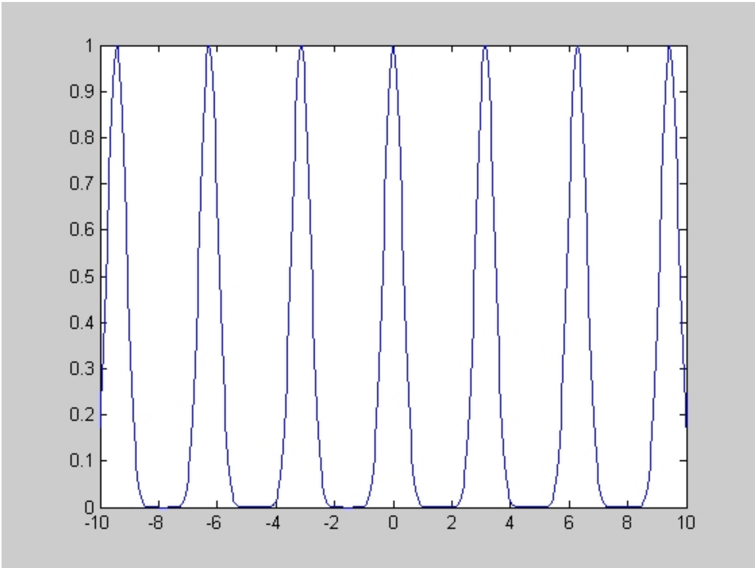
The first is more interesting, because the function isn't real-valued. Furthermore, it depends on the value of Δx, which we don't know. To avoid these problems entirely, we'll just assume that a number n on the vertical axis represents $ni/\Delta x$. Then our second sketch is just a sine curve in the range $[-\pi, \pi]$, which looks like:



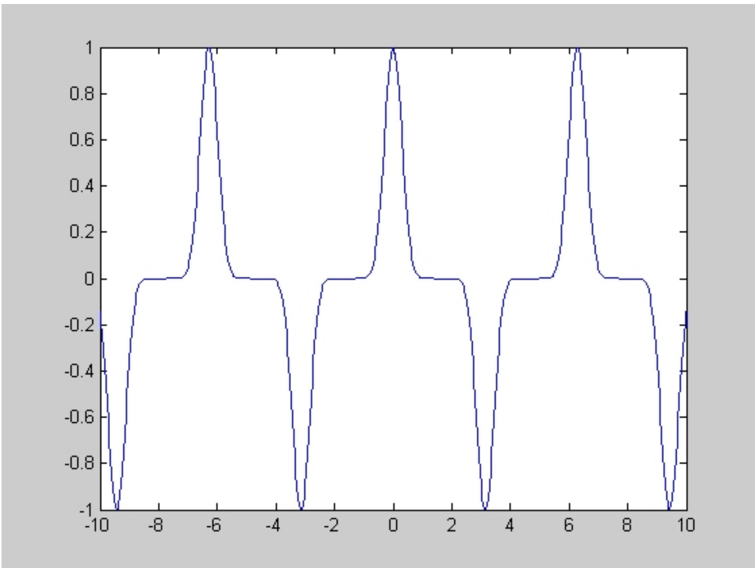
We observe that:

$$\widehat{\mu^m U}/\hat{U} = \cos^m k\Delta x$$

If we sketch this for an odd value of m, we get (using $m = 10$ as an example):



If, on the other hand, we sketch it for an even value of m, we get (using $m = 11$ as an example):



Thus taking successive powers of μ is not a perfect smoothing procedure.

4. The $\ell_2(-\infty,\infty)$ norm of U and the $L_2(-\pi/\Delta x, \pi/\Delta x)$ norm of $\hat{U}$ are defined, respectively, by

$$\|U\|_{\ell_2} = \left(\Delta x \sum_{j=-\infty}^{\infty} |U_j|^2\right)^{1/2}, \quad \|\hat{U}\|_{L_2} = \left(\int_{-\pi/\Delta x}^{\pi/\Delta x} |\hat{U}(k)|^2 \, dk\right)^{1/2}.$$

Prove *Parseval's identity*:

$$\|U\|_{\ell_2} = \frac{1}{\sqrt{2\pi}} \|\hat{U}\|_{L_2}.$$

Answer

We start by observing that:

$$\bar{\hat{U}}(k) = \Delta x \sum_{j=-\infty}^{\infty} \overline{e^{-ikx_j} U_j} = \Delta x \sum_{j=-\infty}^{\infty} e^{ikx_j} \bar{U}_j$$

This follows from the fact that $\overline{e^{i\theta}} = \overline{\cos\theta + i\sin\theta} = \cos\theta - i\sin\theta = e^{-i\theta}$. Now, we want to show that:

$$\|U\|_{\ell_2} = \frac{1}{\sqrt{2\pi}} \|\hat{U}\|_{L_2}.$$

It suffices to show instead that

$$\|U\|_{\ell_2}^2 = \frac{1}{2\pi} \|\hat{U}\|_{L_2}^2,$$

since obviously the norms must be positive. Consider, therefore, the following:

$$\begin{aligned}
\|\hat{U}\|_{L_2}^2 &= \int_{-\pi/\Delta x}^{\pi/\Delta x} |\hat{U}(k)|^2 \, dk \\
&= \int_{-\pi/\Delta x}^{\pi/\Delta x} \hat{U}(k)\bar{\hat{U}}(k) \, dk \qquad \text{since } a\bar{a} = |a|^2 \text{ is an identity} \\
&= \int_{-\pi/\Delta x}^{\pi/\Delta x} \left(\Delta x \sum_{j=-\infty}^{\infty} e^{-ikx_j} U_j\right) \left(\Delta x \sum_{m=-\infty}^{\infty} e^{ikx_m} \bar{U}_m\right) dk \\
&= \sum_{j=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} (\Delta x)^2 U_j \bar{U}_m \int_{-\pi/\Delta x}^{\pi/\Delta x} e^{ik(x_m - x_j)} \, dk \\
&= \sum_{j=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} (\Delta x)^2 U_j \bar{U}_m \int_{-\pi/\Delta x}^{\pi/\Delta x} e^{ik\Delta x(m-j)} \, dk
\end{aligned}$$

Suppose that $m = j$. Then the integral becomes:

$$\int_{-\pi/\Delta x}^{\pi/\Delta x} 1 \; dk = 2\pi/\Delta x$$

Now suppose that $m \neq j$. Then we evaluate the integral to obtain:

$$\left[\frac{e^{ik\Delta x(m-j)}}{i\Delta x(m-j)} \right]_{-\pi/\Delta x}^{\pi/\Delta x}$$

Since $m \neq j$, the denominator of the fraction is non-zero and the numerator is zero. To see the latter, consider that:

$$\begin{array}{lll} & e^{i(\pi/\Delta x)\Delta x(m-j)} - e^{i(-\pi/\Delta x)\Delta x(m-j)} & \\ = & e^{i\pi(m-j)} - e^{-i\pi(m-j)} & \\ = & 2i \sin \pi(m-j) & \text{by the usual identity} \\ = & 0 & \text{since } \sin c\pi = 0 \text{ for integral } c \end{array}$$

So continuing with our original proof, we have that:

$$\|\hat{U}\|_{L_2}^2 = \sum_{j=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} (\Delta x)^2 U_j \bar{U}_m \begin{cases} 0, & \text{if } m \neq j \\ \frac{2\pi}{\Delta x}, & \text{if } m = j \end{cases}$$

From here it's obvious that

$$\|\hat{U}\|_{L_2}^2 = 2\pi\Delta x \sum_{j=-\infty}^{\infty} U_j \bar{U}_j = 2\pi\Delta x \sum_{j=-\infty}^{\infty} |U_j|^2 = 2\pi \|U\|_{\ell_2}^2$$

and our result trivially follows.

5. In the lectures we considered the simplest finite difference approximation of the heat equation $u_t = u_{xx}$, given by

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} = \frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{(\Delta x)^2}, \quad j = \ldots, -2, -1, 0, 1, 2, \ldots; \quad n = 0, 1, 2, \ldots.$$

What would the analogous difference approximation be based on values of U at just every other point in the x direction, i.e. U_{j+2}^n, U_j^n and U_{j-2}^n? Now suppose that you create a new difference approximation from these two schemes by adding $1/2$ of the first difference approximation to $1/2$ of the second difference approximation. Using Fourier analysis, explore how large Δt can be in relation to Δx if this last scheme is to be stable in the $\ell_2(-\infty, \infty)$ norm.

Answer

We derive this using (the by now ubiquitous process of) Taylor series expansion. (Actually this is unnecessary since there's a far easier way, as we'll see in a minute, but we'll do it this way for completeness.) We observe that:

$$\begin{aligned}
& u(x_{j+2}, t_m) + u(x_{j-2}, t_m) \\
= \; & u(x_j + 2\Delta x, t_m) + u(x_j - 2\Delta x, t_m) \\
\approx \; & \left[u(x_j, t_m) + (2\Delta x)\frac{\partial u}{\partial x}(x_j, t_m) + \frac{(2\Delta x)^2}{2}\frac{\partial^2 u}{\partial x^2}(x_j, t_m) \right] + \\
& \left[u(x_j, t_m) + (-2\Delta x)\frac{\partial u}{\partial x}(x_j, t_m) + \frac{(-2\Delta x)^2}{2}\frac{\partial^2 u}{\partial x^2}(x_j, t_m) \right] \\
= \; & 2u(x_j, t_m) + (2\Delta x)^2 \frac{\partial^2 u}{\partial x^2}(x_j, t_m)
\end{aligned}$$

Thus:

$$\frac{\partial^2 u}{\partial x^2}(x_j, t_m) \approx \frac{u(x_{j+2}, t_m) - 2u(x_j, t_m) + u(x_{j-2}, t_m)}{(2\Delta x)^2}$$

This is entirely unsurprising, since it's like the original formula but with Δx replaced with $2\Delta x$. We end up with:

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} = \frac{U_{j+2}^n - 2U_j^n + U_{j-2}^n}{(2\Delta x)^2}$$

From this we create our new difference approximation:

$$\begin{aligned}
\frac{U_j^{n+1} - U_j^n}{\Delta t} &= \frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{2(\Delta x)^2} + \frac{U_{j+2}^n - 2U_j^n + U_{j-2}^n}{2(2\Delta x)^2} \\
&= \frac{U_{j+2}^n + 4U_{j+1}^n - 10U_j^n + 4U_{j-1}^n + U_{j-2}^n}{2(2\Delta x)^2}
\end{aligned}$$

Writing this in matrix notation (for no reason in particular, except perhaps legibility), this becomes:

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} = \frac{1}{8(\Delta x)^2} \begin{bmatrix} 1 & 4 & -10 & 4 & 1 \end{bmatrix} \begin{bmatrix} U_{j+2}^n \\ U_{j+1}^n \\ U_j^n \\ U_{j-1}^n \\ U_{j-2}^n \end{bmatrix}$$

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We now want to explore the stability of this scheme, in particular the range of Δt in relation to Δx such that:

$$\|U^n\|_{\ell_2} \leq \|U^0\|_{\ell_2}, \quad n = 1, \dots, M$$

Well, as per the notes on the stability of the explicit Euler scheme, we start by inserting

$$U_j^n = \frac{1}{2\pi} \int_{-\pi/\Delta x}^{\pi/\Delta x} e^{ikj\Delta x} \hat{U}^n(k)\, dk$$

into the scheme. This gives us:

$$\begin{aligned} & \frac{1}{2\pi} \int_{-\pi/\Delta x}^{\pi/\Delta x} e^{ikj\Delta x} \frac{\hat{U}^{n+1}(k) - \hat{U}^n(k)}{\Delta t}\, dk \\ = \; & \frac{1}{2\pi} \int_{-\pi/\Delta x}^{\pi/\Delta x} e^{ikj\Delta x} \, \frac{e^{2ik\Delta x} + 4e^{ik\Delta x} - 10 + 4e^{-ik\Delta x} + e^{-2ik\Delta x}}{8(\Delta x)^2} \, \hat{U}^n(k)\, dk \end{aligned}$$

Now, by the injectivity of the inverse Fourier transform, F^{-1}, we have that:

$$\frac{\hat{U}^{n+1}(k) - \hat{U}^n(k)}{\Delta t} = \frac{e^{2ik\Delta x} + 4e^{ik\Delta x} - 10 + 4e^{-ik\Delta x} + e^{-2ik\Delta x}}{8(\Delta x)^2} \hat{U}^n(k)$$

Writing $\mu = \frac{\Delta t}{8(\Delta x)^2}$, we have that:

$$\hat{U}^{n+1}(k) = \hat{U}^n(k) + \mu(e^{2ik\Delta x} + 4e^{ik\Delta x} - 10 + 4e^{-ik\Delta x} + e^{-2ik\Delta x})\, \hat{U}^n(k)$$

In other words,

$$\hat{U}^{n+1}(k) = \lambda(k) \hat{U}^n(k),$$

where

$$\lambda(k) = 1 + \mu(e^{2ik\Delta x} + 4e^{ik\Delta x} - 10 + 4e^{-ik\Delta x} + e^{-2ik\Delta x})$$

is the amplication factor.

Now, as per the notes, by Parseval's identity we have that:

$$\begin{aligned}
\|U^{n+1}\|_{\ell_2} &= \frac{1}{\sqrt{2\pi}}\|\hat{U}^{n+1}\|_{L_2} \\
&= \frac{1}{\sqrt{2\pi}}\|\lambda\hat{U}^{n}\|_{L_2} \\
&\leq \frac{1}{\sqrt{2\pi}}\max_k|\lambda(k)|\ \|\hat{U}^{n}\|_{L_2} \\
&= \max_k|\lambda(k)|\ \|U^{n}\|_{\ell_2}
\end{aligned}$$

We're trying to ensure that

$$\|U^{n+1}\|_{\ell_2} \leq \|U^n\|_{\ell_2}, \quad n = 0, 1, \ldots, M-1,$$

so we demand that

$$\max_k|\lambda(k)| \leq 1.$$

This gives us:

$$\begin{aligned}
& \max_k|1+\mu(e^{2ik\Delta x}+4e^{ik\Delta x}-10+4e^{-ik\Delta x}+e^{-2ik\Delta x})| &\leq 1 \\
\Leftrightarrow\ & \max_k|1+\mu(\cos 2k\Delta x+i\sin 2k\Delta x+4(\cos k\Delta x+i\sin k\Delta x)-10 & \\
& \quad +4(\cos k\Delta x-i\sin k\Delta x)+\cos 2k\Delta x-i\sin 2k\Delta x)| &\leq 1 \\
\Leftrightarrow\ & \max_k|1+\mu(2\cos 2k\Delta x+8\cos k\Delta x-10)| &\leq 1 \\
\Leftrightarrow\ & \max_k|1+2\mu(2\cos^2 k\Delta x-1+4\cos k\Delta x-5)| &\leq 1 \\
\Leftrightarrow\ & \max_k|1+4\mu(\cos^2 k\Delta x+2\cos k\Delta x-3)| &\leq 1 \\
\Leftrightarrow\ & \max_k|1+4\mu(\cos k\Delta x+3)(\cos k\Delta x-1)| &\leq 1
\end{aligned}$$

So:

$$-1 \leq 1+4\mu(\cos k\Delta x+3)(\cos k\Delta x-1) \leq 1 \quad \forall k \in [-\pi/\Delta x, \pi/\Delta x]$$

This gives us two conditions, namely

$$4\mu(\cos k\Delta x + 3)(\cos k\Delta x - 1) \leq 0$$

and

$$2 + 4\mu(\cos k\Delta x + 3)(\cos k\Delta x - 1) \geq 0.$$

Well, the first condition can be simplified by observing that $4(\cos k\Delta x + 3)$ is strictly positive (and can thus be divided by without even changing the sign of the inequality), since $\cos k\Delta x \geq -1$, whence:

$$\mu(\cos k\Delta x - 1) \leq 0$$

Since $\mu = \frac{\Delta t}{8(\Delta x)^2}$, we know it must be > 0, so

$$\cos k\Delta x - 1 \leq 0 \Leftrightarrow \cos k\Delta x \leq 1.$$

This was obvious anyway ($\cos$ of anything is ≤ 1), so this condition didn't tell us anything which wasn't already clear. The other condition is more interesting:

$$\mu(\cos k\Delta x + 3)(\cos k\Delta x - 1) \geq -\frac{1}{2}$$

Now $\cos k\Delta x - 1$ varies between -2 and 0, in other words it's ≤ 0 and smallest when it's -2. Let's divide by that, therefore, to get:

$$\mu(\cos k\Delta x + 3) \leq \frac{1}{4}$$

Furthermore, $\cos k\Delta x + 3 > 0$, so we obtain:

$$\mu \leq \frac{1}{4(\cos k\Delta x + 3)}$$

The right-hand side takes its smallest value when its denominator is maximised, i.e. when $\cos k\Delta x = 1$. At that point, the value of the whole right-hand side is $\frac{1}{16}$, so our conclusion is therefore that $\mu \leq \frac{1}{16}$. This means that:

$$\frac{\Delta t}{8(\Delta x)^2} \leq \frac{1}{16}$$

Or in other words:

$$\Delta t \leq \frac{1}{2}(\Delta x)^2$$