Numerical Solution of Differential Equations
Problem Sheet 3

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1. Which of the following would you regard a stiff initial value problem?

   (a) $y' = -(10^5 e^{-10^4 x} + 1)(y-1)$, $y(0) = 2$, on the interval $x \in [0, 1]$. Note that the solution can be found in closed form:

   $$y(x) = e^{10(e^{-10^4 x} - 1)} e^{-x} + 1.$$

   (b)

   $$y_1' = -0.5y_1 + 0.501y_2, \quad y_1(0) = 1.1,$$
   $$y_2' = 0.501y_1 - 0.5y_2, \quad y_2(0) = -0.9,$$

   on the interval $x \in [0, 1]$.

Answer

(a) This is clearly a stiff initial value problem. Problems are called stiff if their solutions decay rapidly towards a common, slowly-varying solution, and the solution of this (which we've helpfully been given) clearly decays rather quickly towards 1.

(b) To determine whether this is stiff or not, we first solve it analytically. Let $\mathbf{y} = (y_1, y_2)^T$ and $\mathbf{y}' = (y_1', y_2')^T$, then:

$$\mathbf{y}' = \underbrace{\begin{bmatrix} -0.5 & 0.501 \\ 0.501 & -0.5 \end{bmatrix}}_{A} \mathbf{y}$$

As per our first-year Calculus course[1], we find the general solution of this by looking at the eigenvalues of A. Accordingly, we write:

$$0 = |A - \lambda I| = \begin{vmatrix} -0.5 - \lambda & 0.501 \\ 0.501 & -0.5 - \lambda \end{vmatrix} = (-0.5 - \lambda)^2 - 0.501^2 = \lambda^2 + \lambda - 0.001001$$

This has roots when $\lambda = 0.001$ or $\lambda = -1.001$, which are therefore the two eigenvalues of A. Calculating the corresponding eigenvectors:

$$\mathbf{0} = (A - 0.001I)\mathbf{x} = \begin{pmatrix} -0.501 & 0.501 \\ 0.501 & -0.501 \end{pmatrix} \mathbf{x}$$

So if $\mathbf{x} = (x_1, x_2)^T$, then $-0.501x_1 + 0.501x_2 = 0 \Rightarrow x_1 = x_2$, whence any multiple of $(1, 1)^T$ is an eigenvector of A corresponding to $\lambda = 0.001$. Similarly:

$$\mathbf{0} = (A + 1.001I)\mathbf{x} = \begin{pmatrix} 0.501 & 0.501 \\ 0.501 & 0.501 \end{pmatrix} \mathbf{x}$$

From this it's quite clear that the eigenvectors corresponding to the other eigenvalue are multiples of $(1, -1)^T$.

Our general solution is therefore given by:

$$\mathbf{y} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{0.001x} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-1.001x}$$

[1] Which was sadly rather too distant a memory!

In other words, $y_1 = c_1 e^{0.001x} + c_2 e^{-1.001x}$ and $y_2 = c_1 e^{0.001x} - c_2 e^{-1.001x}$. We now apply the initial conditions: if $y_1(0) = 1.1$, then $c_1 + c_2 = 1.1$, and if $y_2(0) = -0.9$, then $c_1 - c_2 = -0.9$. So $2c_1 = 0.2 \Rightarrow c_1 = 0.1$ and $c_2 = 1$. Whence $y_1 = 0.1e^{0.001x} + e^{-1.001x}$ and $y_2 = 0.1e^{0.001x} - e^{-1.001x}$.

This shouldn't be regarded as a stiff problem. As $x \to \infty$, the $e^{-1.001x}$ terms in both y_1 and y_2 will tend to 0, leaving y_1 and y_2 to grow exponentially like $0.1e^{0.001x}$. Exponential growth cannot be described as a 'slowly-varying solution', so the problem is not stiff.

2. Consider the θ-method

$$y_{n+1} = y_n + h\left[(1-\theta)f_n + \theta f_{n+1}\right]$$

for $\theta \in [0, 1]$.

(a) Show that the method is A-stable for $\theta \in [1/2, 1]$.

(b) A method is said to be $A(\alpha)$-stable, $\alpha \in (0, \pi/2)$, if its region of absolute stability (as a set in the complex plane), contains the infinite wedge $\{\bar{h} : \pi - \alpha < \arg(\bar{h}) < \pi + \alpha\}$. Find all $\theta \in [0, 1]$ such that the θ-method is $A(\alpha)$-stable for some $\alpha \in (0, \pi/2)$.

Answer

(a) We apply the θ-method to $y' = \lambda y$, $y(0) = 1$, where $Re(\lambda) < 0$. We need to find its region of absolute stability (for various values of θ, obviously). As per the previous sheet, we first work out what its stability polynomial is. So, we note that (referring to the general definition of a linear multistep method), $\alpha_1 = 1$, $\alpha_0 = -1$, $\beta_1 = \theta$ and $\beta_0 = 1 - \theta$, whence:

$$\rho(z) = z - 1$$

and:

$$\sigma(z) = \theta z + (1-\theta) = \theta(z-1) + 1$$

Hence the stability polynomial can be calculated as:

$$\pi(z; \bar{h}) = \rho(z) - \bar{h}\sigma(z) = (z-1) - \bar{h}(\theta(z-1)+1) = (z-1)(1-\bar{h}\theta) - \bar{h}$$

where $\bar{h} = \lambda h$. For absolute stability, we require that the root z of this polynomial satisfies $|z| < 1$, so:

$$\left|\frac{\bar{h}}{1-\bar{h}\theta} + 1\right| = \frac{|\bar{h}(1-\theta)+1|}{|1-\bar{h}\theta|} < 1$$

So:

$$|\bar{h}(1-\theta) + 1| < |1 - \bar{h}\theta|$$

Now, since $\bar{h} \in \mathbb{C}$ we write $\bar{h} = x + iy$ and calculate things that way, as the complex numbers are not ordered so we can't proceed by a more direct approach. We get:

$$\begin{aligned}
|(x+iy)(1-\theta)+1| &< |1-(x+iy)\theta| \\
|[x(1-\theta)+1]+i[y(1-\theta)]| &< |[1-x\theta]+i[-y\theta]| \\
x^2(1-\theta)^2+2x(1-\theta)+1+y^2(1-\theta)^2 &< 1-2x\theta+x^2\theta^2+y^2\theta^2 \\
(x^2+y^2)(1-\theta)^2+2x(1-\theta)+1 &< 1-2x\theta+(x^2+y^2)\theta^2 \\
(x^2+y^2)(1-2\theta)+2x &< 0
\end{aligned}$$

Now, if $\theta \in [1/2, 1]$, then $-1 \leq 1-2\theta \leq 0$, so we have:

$$(x^2+y^2)(1-2\theta)+2x \leq 2x < 0$$

i.e. $x < 0$. But $x \equiv Re(\bar{h})$, so this condition is satisfied for *all* $\bar{h}$ in the left-hand complex half-plane, so the method is A-stable for $\theta \in [1/2, 1]$.

(b) We observe that our region of absolute stability for the θ-method (with a particular value of θ) was calculated above to be:

$$\{\bar{h} : |\bar{h}|^2(1-2\theta)+2Re(\bar{h}) < 0\}$$

Alternatively, we can write this as:

$$\{\bar{h} : Re(\bar{h}) < \bar{h}^2(\theta - 1/2)\} \quad (*)$$

(Incidentally, this makes it clear where the above answer came from: clearly if $Re(\bar{h}) < 0$, then the inequality above is always satisfied provided $\theta - 1/2 \geq 0$, i.e. provided $\theta \geq 1/2$.)

For a start, it's clear that for $\theta \in [1/2, 1]$, the θ-method is $A(\alpha)$-stable for all $\alpha \in (0, \pi/2)$, since the infinite wedges are all contained in the left-hand complex half-plane and we just showed in part (a) that the regions of absolute stability for those values of θ included the *whole* left-hand complex half-plane, never mind any particular infinite wedge contained within it.

For the rest, we note[2] that iff the θ-method (for some value of θ) is $A(\alpha)$-stable for *any* $\alpha \in (0, \pi/2)$ then the method's region of absolute stability (for that value of θ) contains the negative real axis. Or to put it another way, if, for some value of θ, the set defined by $(*)$ contains all negative real numbers, then the θ-method *with that value of* θ is $A(\alpha)$-stable for *some* value of $\alpha \in (0, \pi/2)$, even though we don't necessarily know which value that is. So we want to find the values of $\theta \in [0, 1]$ s.t. for every real $x < 0$,

$$x^2(\theta - 1/2) - x > 0.$$

[2]After reading through the extended lecture notes, at any rate!

Well, since $x < 0$, we can divide through by it (making sure to change the sign, of course!), to give:

$$x(\theta - 1/2) - 1 < 0,$$

from which we get (again switching the sign, because $x < 0$):

$$\theta > 1/x + 1/2$$

In the limit, as $x \to -\infty$, this condition becomes $\theta \geq 1/2$ (note that it's not $\theta > 1/2$, because we can make $1/x$ arbitrarily small). So we can see that no value of $\theta < 1/2$ will give rise to a θ-method that is $A(\alpha)$-stable.

To make things even clearer, fix some value $\hat{\theta} < 1/2$, then considering the set

$$S = \{\bar{h} : Re(\bar{h}) < \bar{h}^2(\hat{\theta} - 1/2)\}$$

and the case where $\bar{h} \equiv x$ is real and negative, we see that if $x \leq 1/(\hat{\theta} - 1/2)$, then $x \notin S$. Since there is always some negative real satisfying this, the negative real axis can't be contained in S when $\theta < 1/2$.

Note: In the next question you will find it helpful to exploit the following result, known as *Schur's criterion*. Consider the polynomail $\phi(z) = c_k z^k + \ldots + c_1 z + c_0$, $c_k \neq 0$, $c_0 \neq 0$, with complex coefficients. The polynomial ϕ is said to be a *Schur polynomial* if each of its roots z_j satisfies $|z_j| < 1$, $j = 1, \ldots, k$. Given the polynomial $\phi(z)$, as above, consider the polynomial

$$\hat{\phi}(z) = \bar{c}_0 z^k + \bar{c}_1 z^{k-1} + \ldots + \bar{c}_{k-1} z + \bar{c}_k,$$

where $\bar{c}_j$ denotes the complex conjugate of c_j, $j = 1, \ldots, k$. Further, let us define

$$\phi_1(z) = \frac{1}{z}\left[\hat{\phi}(0)\phi(z) - \phi(0)\hat{\phi}(z)\right].$$

Clearly ϕ_1 has degree $\leq k-1$. The polynomial ϕ is a Schur polynomial if and only if $|\hat{\phi}(0)| > |\phi(0)|$ and ϕ_1 is a Schur polynomial.

3. Show that the second-order backward differentiation method

$$3y_{n+2} - 4y_{n+1} + y_n = 2hf(x_{n+2}, y_{n+2})$$

is A-stable.

Answer

To determine the method's A-stability, we first of all need to determine its stability polynomial. Again referring to the general definition of a linear multistep method, we have that $\alpha_2 = 3$, $\alpha_1 = -4$, $\alpha_0 = 1$, $\beta_2 = 2$ and $\beta_1 = \beta_0 = 0$. Thus:

$$\rho(z) = 3z^2 - 4z + 1$$

$$\sigma(z) = 2z^2$$

$$\pi(z; \bar{h}) = \rho(z) - \bar{h}\sigma(z) = 3z^2 - 4z + 1 - \bar{h}(2z^2) = z^2(3 - 2\bar{h}) + z(-4) + 1$$

We want to use Schur's criterion to find the range of $\bar{h} \in \mathbb{C}$ such that $\pi(z; \bar{h})$ is a Schur polynomial. If, as we hope, it includes the whole negative half-space, then we will have shown that the method is A-stable, as required. Accordingly, we calculate that

$$\hat{\pi}(z; \bar{h}) = z^2 + z(-4) + (3 - 2\hat{\bar{h}}),$$

where $\hat{\bar{h}}$ is the complex conjugate of $\bar{h}$ (the notation is rather obtuse, but it will have to do).

Furthermore, we define:

$$\pi_1(z; \bar{h}) = \frac{1}{z}\left[\hat{\pi}(0; \bar{h})\pi(z; \bar{h}) - \pi(0; \bar{h})\hat{\pi}(z; \bar{h})\right].$$

We need to check both that $|\hat{\pi}(0;\bar{h})| > |\pi(0;\bar{h})|$ and that π_1 is a Schur polynomial (we only need to worry about $\bar{h}$ in the negative half-plane, incidentally). Considering the inequality condition first:

$$|\hat{\pi}(0;\bar{h})| = |3 - 2\hat{\bar{h}}|$$
$$|\pi(0;\bar{h})| = 1$$

If we let $\bar{h} = x + iy$, then $\hat{\bar{h}} = x - iy$ and $|3 - 2\hat{\bar{h}}| = |3 - 2(x - iy)|$. It suffices to determine when:

$$|3 - 2(x - iy)|^2 > 1^2$$

We therefore calculate as follows:

$$|3 - 2(x - iy)|^2 = |(3 - 2x) + i(2y)|^2 = (3 - 2x)^2 + 4y^2 = 9 - 12x + 4(x^2 + y^2)$$

This is greater than 1 provided:

$$4(x^2 + y^2) - 12x + 8 > 0$$

Well:

$$4(x^2 + y^2) - 12x + 8 \geq 4x^2 - 12x + 8 = 4(x^2 - 3x + 2) = 4(x - 2)(x - 1)$$

In other words, the roots of this are in the positive half-space, so we don't need to worry about them (note that changing y would just move the quadratic up a bit and the roots would still be in the positive half-space). Thus when $x < 0$, i.e. when $\bar{h}$ is in the negative half-space of the complex plane, $|\hat{\pi}(0;\bar{h})| > |\pi(0;\bar{h})|$.

Now for the condition on π_1. We calculate that:

$$\begin{aligned}
\pi_1(z;\bar{h}) &= \frac{1}{z}\left[\hat{\pi}(0;\bar{h})\pi(z;\bar{h}) - \pi(0;\bar{h})\hat{\pi}(z;\bar{h})\right] \\
&= \frac{1}{z}\left[(3 - 2\hat{\bar{h}})(z^2(3 - 2\bar{h}) - 4z + 1) - (z^2 - 4z + (3 - 2\hat{\bar{h}}))\right] \\
&= \frac{1}{z}\left[z^2((3 - 2\hat{\bar{h}})(3 - 2\bar{h}) - 1) - 4z((3 - 2\hat{\bar{h}}) - 1)\right] \\
&= z((3 - 2\hat{\bar{h}})(3 - 2\bar{h}) - 1) - 4((3 - 2\hat{\bar{h}}) - 1) \\
&= z((3 - 2(x - iy))(3 - 2(x + iy)) - 1) - 4((3 - 2(x - iy)) - 1) \\
&= z((3 - 2x)^2 + 4y^2 - 1) + 8((x - 1) - iy) \\
&= z(9 - 12x + 4(x^2 + y^2) - 1) + 8((x - 1) - iy) \\
&= z(8 - 12x + 4(x^2 + y^2)) + 8((x - 1) - iy) \\
&= 4z(2 - 3x + x^2 + y^2) + 8((x - 1) - iy) \\
&= 4z((x - 2)(x - 1) + y^2) + 8((x - 1) - iy)
\end{aligned}$$

This has a root when:

$$z = \frac{-8((x-1)-iy)}{4((x-2)(x-1)+y^2)} = \frac{-2((x-1)-iy)}{(x-2)(x-1)+y^2}$$

We want to show that $|z| < 1$ provided $x < 0$, i.e. that:

$$\left| \frac{(x-1)-iy}{(x-2)(x-1)+y^2} \right| < \frac{1}{2}$$

Well, firstly split this into real and imaginary parts:

$$\left| \underbrace{\frac{x-1}{(x-2)(x-1)+y^2}}_{A} + i \underbrace{\frac{-y}{(x-2)(x-1)+y^2}}_{B} \right|$$

We note that

$$|A| \leq \frac{1}{|x-2|}$$

(since adding y^2 can only increase the denominator when $x < 0$) and that for $x < 0$, the denominator of the above is always strictly greater than 2, whence $|A| < \frac{1}{2}$. This already gives us that the method is $A(0)$-stable, since the region of absolute stability includes everywhere where $y = 0$, i.e. the whole negative real axis.

To show that it's also A-stable, we need to show that $A^2 + B^2 < \frac{1}{4}$. (This is clearly true when $B = 0$, since then $A^2 < \frac{1}{2}^2 = \frac{1}{4}$.) We therefore calculate:

$$A^2 + B^2 = \frac{(x-1)^2 + y^2}{[(x-2)(x-1)+y^2]^2} = \frac{(x-1)^2 + y^2}{(x-2)^2(x-1)^2 + 2(x-2)(x-1)y^2 + y^4}$$

To show that this is less than $\frac{1}{4}$, it suffices to show that (when $x < 0$):

$$4((x-1)^2 + y^2) < (x-2)^2(x-1)^2 + 2(x-2)(x-1)y^2 + y^4$$

In other words, we must show that:

$$y^4 + y^2(\underbrace{2(x-2)(x-1) - 4}_{C}) + \underbrace{(x-2)^2(x-1)^2 - 4(x-1)^2}_{D} > 0$$

We know that $y^4 > 0$, so it suffices to show that $C > 0$ and $D > 0$. Well, if $x < 0$, then $C > 0$, for $(x-2)(x-1) > 2$. Furthermore, $D > 0$, since:

$$D = \underbrace{(x-1)^2}_{E} \underbrace{((x-2)^2 - 4)}_{F}$$

and $E > 0$ because it's a square and $F > 0$ because $x < 0$ and hence $(x-2)^2 > 4$. So $A^2 + B^2 < \frac{1}{4}$ as desired, and if we backtrack, we have what we want. So the method is A-stable. (Finally!)