Numerical Solution of Differential Equations
Problem Sheet 1

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1. Verify that the following functions satisfy a Lipschitz condition with respect to *y*, uniformly in *x*, on the respective intervals:

   (a) $f(x,y) = 2yx^{-4}$, $x \in [1,\infty)$, $y \in \mathbb{R}$;

   (b) $f(x,y) = e^{-x^2}\ \tan^{-1} y$, $x \in [1,\infty)$, $y \in \mathbb{R}$;

   (c) $f(x,y) = 2y(1+y^2)^{-1}(1+e^{-|x|})$, $x \in \mathbb{R}$, $y \in \mathbb{R}$.

Answer

(a) Let $R = \{(x,y) \in \mathbb{R}^2 : 1 \le x < \infty\}$. We are required to show that

$$\exists L > 0 \ \cdot\ \forall (x,y_1),\ (x,y_2) \in R \ \cdot\ |f(x,y_1) - f(x,y_2)| \le L\,|y_1 - y_2|$$

This requirement becomes

$$\exists L > 0 \ \cdot\ \forall (x,y_1),\ (x,y_2) \in R \ \cdot\ \left|2y_1x^{-4} - 2y_2x^{-4}\right| \le L\,|y_1 - y_2|$$

and then

$$\exists L > 0 \ \cdot\ \forall (x,y_1),\ (x,y_2) \in R \ \cdot\ \left|2x^{-4}\,(y_1 - y_2)\right| \le L\,|y_1 - y_2|$$

Now for any a, b, we have that $|ab| = |a||b|$, so $|2x^{-4}\,(y_1 - y_2)| = |2x^{-4}|\,|y_1 - y_2|$. Thus, provided $y_1 \ne y_2$, we only have to find an L s.t. for any value of $x \in [1,\infty)$, $L \ge |2x^{-4}| = 2x^{-4}$, the last equality following since x is positive (if $y_1 = y_2$, the condition is trivially satisfied as $0 \le 0$: any value of L will work). Well, $2x^{-4}$ is at its largest in the range $[1,\infty)$ when $x = 1$, so if we choose any $L \ge 2$ then the condition will be satisfied.

(b) Let R be as in the previous part. We are required to show that

$$\exists L > 0 \ \cdot\ \forall (x,y_1),\ (x,y_2) \in R \ \cdot\ |e^{-x^2}\left(\tan^{-1} y_1 - \tan^{-1} y_2\right)| \le L|y_1 - y_2|$$

In other words, provided $y_1 \ne y_2$ (if $y_1 = y_2$ then any value of L will work, as in the previous part), we need to show that

$$\exists L > 0 \ \cdot\ \forall (x,y_1),\ (x,y_2) \in R \ \cdot\ L \ge \frac{|e^{-x^2}||\tan^{-1} y_1 - \tan^{-1} y_2|}{|y_1 - y_2|}$$

Now we saw in the lecture notes that $|\tan^{-1} y_1 - \tan^{-1} y_2| \le |y_1 - y_2|$, so we know that

$$\frac{|\tan^{-1} y_1 - \tan^{-1} y_2|}{|y_1 - y_2|} \le 1$$

So given that that fraction will be at most 1, it suffices to find an L s.t. for any value of $x \in [1,\infty)$, $L \ge |e^{-x^2}| = e^{-x^2}$. Since e^{-x^2} has its greatest value when $x = 1$, any $L \ge e^{-1}$ will do.

(c) Let $R = \mathbb{R}^2$. We are required to show that

$$\exists L > 0 \ \cdot \ \forall (x, y_1),\ (x, y_2) \in R \ \cdot \ \left| 2\left(1 + e^{-|x|}\right) \left(\frac{y_1}{1 + y_1^2} - \frac{y_2}{1 + y_2^2} \right) \right| \leq L|y_1 - y_2|$$

In other words, we must show that

$$\exists L > 0 \ \cdot \ \forall (x, y_1),\ (x, y_2) \in R \ \cdot \ 2\left|1 + e^{-|x|}\right| \left| \frac{y_1}{1 + y_1^2} - \frac{y_2}{1 + y_2^2} \right| \leq L|y_1 - y_2|$$

Now, we observe that

$$\left| \frac{y_1}{1 + y_1^2} - \frac{y_2}{1 + y_2^2} \right| \leq |y_1 - y_2|$$

To show this, assume w.l.o.g. that $1 + y_1^2 \geq 1 + y_2^2$, then

$$\left| \frac{y_1}{1 + y_1^2} - \frac{y_2}{1 + y_2^2} \right| \leq \left| \frac{y_1 - y_2}{1 + y_1^2} \right| = \frac{|y_1 - y_2|}{1 + y_1^2} \leq |y_1 - y_2|$$

So we only have to find an L s.t. for any value of $x \in \mathbb{R}$, $L \geq 2\left|1 + e^{-|x|}\right|$. The right-hand side assumes its greatest value when $x = 0$, so we can choose any $L \geq 2\left|1 + e^{-0}\right| = 4$.

2. Suppose that *m* is a fixed positive integer. Show that the initial value problem

$$y' = y^{2m/(2m+1)},\ y(0) = 0$$

has infinitely many continuously differentiable solutions. Why does this not contradict Picard's Theorem?

Answer

This is the general case of the one we saw in lectures. Clearly $y = 0$ is one solution. We can find another solution by separation of variables:

$$\frac{dy}{dx} = y^{2m/(2m+1)}$$

$$\int_0^y y^{-2m/(2m+1)} dy = \int_0^x dx$$

$$\left[(2m + 1)y^{1/(2m+1)}\right]_0^y = [x]_0^x$$

$$(2m + 1)y^{1/(2m+1)} = x$$

$$y = \left(\frac{x}{2m + 1}\right)^{2m+1}$$

As was shown in the lectures, we can also get infinitely many other solutions by combining the two:

$$y_c(x) = \begin{cases} 0 & 0 \le x \le c \\ \left(\dfrac{x-c}{2m+1}\right)^{2m+1} & c \le x < \infty \end{cases} \qquad \forall c > 0$$

So the i.v.p. has infinitely many solutions and they're clearly all continuously differentiable, as required.

This doesn't contradict Picard's Theorem because $f(x,y) = y^{2m/(2m+1)}$ doesn't meet the Lipschitz condition required for Picard's Theorem to apply. We will show this, as per the lecture notes, as follows:

For some $h > 0$ and $k > 0$, let

$$R = \left\{(x,y) \in \mathbb{R}^2 : |x| \le h, |y| \le k\right\}$$

Then $f(x,y) = y^{2m/(2m+1)}$ violates the Lipschitz condition in R for any $h > 0$ and $k > 0$. Indeed, for any $L > 0$, there exists y, $0 < |y| < (1/L)^{2m+1}$, s.t.

$$|f(x,y) - f(x,0)| = |y^{2m/(2m+1)}| > L|y|$$

We thus conclude that $f(x,y) = y^{2m/(2m+1)}$ violates the Lipschitz condition in a neighbourhood R of $(0,0)$ for any choice of $h > 0$ and $k > 0$.

3. Van der Pol's equation

$$y'' - \varepsilon(1 - y^2)y' + y = 0$$

subject to the initial conditions $y(a) = A_1$ and $y'(a) = A_2$, where A_1 and A_2 are given real numbers, and $\varepsilon > 0$ a parameter, models electrical circuits connected with electronic oscillators. Rewrite the equation as a coupled system of two first-order differential equations with appropriate initial conditions. Formulate Euler's method for this system, when $\varepsilon = 1$, $a = 0$, $A_1 = 1/2$ and $A_2 = 1/2$, on the interval $[0, 1]$ using a mesh of uniform spacing h. Compute the Euler approximations to $y(x)$ and $y'(x)$ at the point $x = h$.

Answer

Letting $z = y'$, whence $z' = y''$, we obtain the first-order coupled system

$y' = z$
$z' = \varepsilon(1 - y^2)z - y$

where the initial conditions are $y(a) = A_1$ and $z(a) = A_2$.

Letting $\mathbf{Y}(x) = \begin{pmatrix} y(x) \\ z(x) \end{pmatrix}$, we formulate Euler's method as follows:

$$\mathbf{Y}_{n+1} = \mathbf{Y}_n + hf(x_n, \mathbf{Y}_n)$$

In this we note that if we write $\mathbf{Y}_n = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, the above definitions give us that:

$$f(x_n, \mathbf{Y}_n) = \mathbf{Y}_n' = \begin{pmatrix} y_2 \\ \varepsilon(1 - y_1^2)y_2 - y_1 \end{pmatrix}$$

We take as the initial condition:

$$\mathbf{Y}_a = \begin{pmatrix} y(a) \\ z(a) \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$$

Plugging in the values we're given, we have:

$$\mathbf{Y}_0 = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

And:

$$\mathbf{Y}_{n+1} = \begin{pmatrix} y_1 + hy_2 \\ y_2 + h((1 - y_1^2)y_2 - y_1) \end{pmatrix} \text{ where } \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \mathbf{Y}_n$$

To compute the approximations to $y(x)$ and $y'(x)$ at $x = h$, we simply have to evaluate $\mathbf{Y}_1$:

$$\mathbf{Y}_1 = \begin{pmatrix} \frac{1}{2} + h(\frac{1}{2}) \\ \frac{1}{2} + h((1 - \frac{1}{2}^2) \times \frac{1}{2} - \frac{1}{2}) \end{pmatrix}$$

So $y(h) \approx \frac{1}{2}(1 + h)$ and $y'(h) \approx \frac{1}{2} - \frac{h}{8}$.

4. Consider the initial value problem

$$y' = \log\log(4+y^2),\ x \in [0,1],\ y(0) = 1,$$

and the sequence $(y_n)_{n=0}^N$, $N \geq 1$, generated by the explicit Euler method

$$y_{n+1} = y_n + h\log\log(4+y_n^2),\ n = 0,\ldots,N-1,\ y_0 = 1,$$

using the mesh points $x_n = nh$, $n = 0,\ldots,N$, with spacing $h = 1/N$. Here log denotes the logarithm with base e.

(a) Let T_n denote the truncation error of Euler's method at $x = x_n$ for this initial value problem. Show that $|T_n| \leq h/(4e)$.

(b) Verify that

$$|y(x_{n+1}) - y_{n+1}| \leq (1+hL)\,|y(x_n) - y_n| + h\,|T_n|\,,\ n = 0,\ldots,N-1,$$

where $L = 1/(2\log 4)$.

(c) Find a positive integer N_0, as small as possible, such that

$$\max_{0\leq n\leq N} |y(x_n) - y_n| \leq 10^{-4}$$

whenever $N \geq N_0$.

Answer

(a) We observe from the notes that:

$$T_n = \frac{1}{2}hy''(\zeta_n),\ \zeta_n \in [x_n, x_{n+1}]$$

Thus:

$$|T_n| = \left|\frac{1}{2}hy''(\zeta_n)\right|,\ \zeta_n \in [x_n, x_{n+1}]$$

It therefore suffices to show that $|y''(\zeta_n)| \leq |\frac{1}{2e}|$, since then $|T_n| \leq \left|\frac{1}{2}h\frac{1}{2e}\right| = \frac{h}{4e}$ as required.
To do this, we first need to work out what $|y''|$ is:

$$|y''| = \left|\frac{1}{\log(4+y^2)}\cdot\frac{1}{4+y^2}\cdot 2yy'\right| = \underbrace{\left|\frac{\log\log(4+y^2)}{\log(4+y^2)}\right|}_{A}\cdot\underbrace{\left|\frac{2y}{4+y^2}\right|}_{B}$$

Now, B has a maximum where $y=2$, when it equals $\frac{4}{8} = \frac{1}{2}$.
What about A? Well, consider differentiating $\frac{\log\log u}{\log u}$ w.r.t. u to give:

$$\frac{\log u\cdot\frac{1}{\log u}\cdot\frac{1}{u} - \log\log u\cdot\frac{1}{u}}{(\log u)^2} = \frac{\frac{1}{u}(1-\log\log u)}{(\log u)^2}$$

Equating this to zero to find the turning points, we observe that it is zero if and only if $1 - \log\log u = 0$, i.e. iff $u = e^e$. So A attains its greatest value when $4 + y^2 = e^e$. This greatest value turns out to be $\frac{1}{e}$, which is just what we need.
We finish the proof by observing that:

$$|y''(\zeta_n)| \leq \max|y''| = AB = \frac{1}{2e}$$

(b) From the definitions of $y(x_{n+1})$ and y_{n+1} (see the bit in the notes on using the truncation error to bound the global error), we calculate that:

$$\begin{aligned}
& |y(x_{n+1}) - y_{n+1}| \\
= \ & |y(x_n) + hf(x_n, y(x_n)) + hT_n - y_n - hf(x_n, y_n)| \\
= \ & |y(x_n) - y_n + h[(f(x_n, y(x_n)) - f(x_n, y_n)) + T_n]| \\
\leq \ & |y(x_n) - y_n + h[L|y(x_n) - y_n| + T_n]| \qquad \{\text{by the Lipschitz condition}\} \\
= \ & (1 + hL)|y(x_n) - y_n| + h|T_n|
\end{aligned}$$

We have thus derived the inequality in the question; it remains to show that the constant L in this case is $1/(2\log 4)$. In other words, we must determine the Lipschitz constant for the function $f(x, y) = \log\log(4 + y^2)$. Rather than using the method I used in Question 1, I'll do it the proper way this time. We note that:

$$\frac{\partial f(x, y)}{\partial y} = \underbrace{\frac{1}{\log(4 + y^2)}}_{A'} \cdot \underbrace{\frac{2y}{4 + y^2}}_{B'}$$

Now $B' \leq \frac{1}{2}$, as in part (a), and $A' \leq \frac{1}{\log 4}$, so $L = A'B' = 1/(2\log 4)$ works as a Lipschitz constant here.

(c) For brevity of notation, we define:

$$E_n \stackrel{def}{=} |y(x_n) - y_n|$$

So from (b) we observe that:

$$E_{n+1} \leq (1 + hL)E_n + h|T_n|$$

Noting that $E_0 = |y(x_0) - y_0| = |1 - 1| = 0$, we can proceed to inductively find a bound for E_n:

$$
\begin{array}{lcl}
E_1 & \leq & (1+hL)E_0 + h|T_0| = h|T_0| \\
E_2 & \leq & (1+hL)E_1 + h|T_1| = (1+hL)h|T_0| + h|T_1| \\
E_3 & \leq & (1+hL)E_2 + h|T_2| = (1+hL)^2 h|T_0| + (1+hL)h|T_1| + h|T_2| \\
E_4 & \leq & (1+hL)^3 h|T_0| + (1+hL)^2 h|T_1| + (1+hL)h|T_2| + h|T_3| \\
\ldots & & \\
E_n & \leq & h\left[\sum_{i=0}^{n-1}(1+hL)^{n-1-i}|T_i|\right] \\
 & \leq & h\underbrace{\left(\max_{0\leq k\leq n-1}|T_k|\right)}_{T_{max}}\left[\sum_{i=0}^{n-1}(1+hL)^i\right] \\
 & = & hT_{max}\dfrac{(1+hL)^n - 1}{(1+hL)-1} \\
 & = & \dfrac{T_{max}}{L}\left[(1+hL)^n - 1\right] \\
 & \leq & \dfrac{T_{max}}{L}\left[\left(\displaystyle\sum_{i=0}^{\infty}\dfrac{(hL)^i}{i!}\right)^n - 1\right] \\
 & = & \dfrac{T_{max}}{L}\left[e^{nhL} - 1\right] \\
 & = & \dfrac{T_{max}}{L}\left[e^{(x_n - 0)L} - 1\right] \\
 & = & \dfrac{T_{max}}{L}\left[e^{x_n L} - 1\right]
\end{array}
$$

This is starting to look quite promising. We use our results from (a) and (b) and the fact that $x_n \in [0,1]$ (and thus $x_n \leq 1$) to observe that:

$$
E_n \leq \frac{2\log 4}{4e}\left[e^{1/(2\log 4)} - 1\right] \times h = 0.1107429961h
$$

This is true for all $n = 0, 1, \ldots, N$, so if we now simply require that $E_n \leq 10^{-4}$, we will obtain our answer:

$$
\begin{array}{lrcl}
 & 0.1107429961h & \leq & 10^{-4} \\
\Leftrightarrow & 0.1107429961\left(\dfrac{1}{N}\right) & \leq & 10^{-4} \\
\Leftrightarrow & N & \geq & 1107.429961 \\
\Leftrightarrow & N & \geq & 1108 = N_0
\end{array}
$$