Intelligent Systems II
Exercise Sheet 4

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1. (a) This is a relative velocity problem of the kind done at A-Level.[1] We observe that at time t the ball is at $\mathbf{a} + t\mathbf{u}$ and the robot is at $t\mathbf{v}$ for some velocity $\mathbf{v}$ such that $|\mathbf{v}| = S$. If they are to intercept, then at some time t we must have $\mathbf{a} + t\mathbf{u} = t\mathbf{v}$, i.e. $\mathbf{a} = t(\mathbf{v} - \mathbf{u})$. So the direction of $\mathbf{v} - \mathbf{u}$ is the same as that of $\mathbf{a}$. (This is a special case of the more general result that if one object is to intercept another then the relative velocity involved must be along the line joining their start points.)

Let's write $\mathbf{v} - \mathbf{u} = k\mathbf{a}$ (where $k = \frac{1}{t}$), so $\mathbf{v} = k\mathbf{a} + \mathbf{u}$. We know that $|\mathbf{v}| = S$, so $|k\mathbf{a} + \mathbf{u}| = S$. Whence:

$$(ka_x + u_x)^2 + (ka_y + u_y)^2 = k^2a_x^2 + 2ka_xu_x + u_x^2 + k^2a_y^2 + 2ka_yu_y + u_y^2 = S^2$$

This gives us a quadratic we can solve for k:

$$k^2(a_x^2 + a_y^2) + k(2(a_xu_x + a_yu_y)) + (u_x^2 + u_y^2 - S^2) = 0$$

This is just:

$$k^2|\mathbf{a}|^2 + k(2\mathbf{a} \bullet \mathbf{u}) + (|\mathbf{u}|^2 - S^2) = 0$$

So:

$$\begin{aligned} k &= \frac{-2\mathbf{a} \bullet \mathbf{u} \pm \sqrt{4(\mathbf{a} \bullet \mathbf{u})^2 - 4|\mathbf{a}|^2(|\mathbf{u}|^2 - S^2)}}{2|\mathbf{a}|^2} \\ &= \frac{-\mathbf{a} \bullet \mathbf{u} \pm \sqrt{(\mathbf{a} \bullet \mathbf{u})^2 - |\mathbf{a}|^2(|\mathbf{u}|^2 - S^2)}}{|\mathbf{a}|^2} \\ &= \frac{-\mathbf{a} \bullet \mathbf{u} \pm \sqrt{(\mathbf{a} \bullet \mathbf{u})^2 + |\mathbf{a}|^2(S^2 - |\mathbf{u}|^2)}}{|\mathbf{a}|^2} \\ &= \frac{\frac{-\mathbf{u} \bullet \mathbf{a}}{|\mathbf{a}|} \pm \sqrt{S^2 - \left(|\mathbf{u}|^2 - \left(\frac{\mathbf{u} \bullet \mathbf{a}}{|\mathbf{a}|}\right)^2\right)}}{|\mathbf{a}|} \end{aligned}$$

Observe that for there to be a solution:

$$S^2 - \left(|\mathbf{u}|^2 - \left(\frac{\mathbf{u} \bullet \mathbf{a}}{|\mathbf{a}|}\right)^2\right) \geq 0$$

In other words:

$$S \geq (+)\sqrt{|\mathbf{u}|^2 - \left(\frac{\mathbf{u} \bullet \mathbf{a}}{|\mathbf{a}|}\right)^2}$$

This makes perfect geometric sense. Consider the following: $\dfrac{\mathbf{u} \bullet \mathbf{a}}{|\mathbf{a}|}$ is the component of $\mathbf{u}$ in the direction of $\mathbf{a} - \mathbf{0}$, i.e. the initial vector from the robot to the ball. The RHS of the above inequality is then the component of $\mathbf{u}$ in the perpendicular to $\mathbf{a} - \mathbf{0}$. Now clearly if S isn't at least as big as this, then the ball will move in the perpendicular direction faster than the robot can move to catch it. (Think about it as follows: the robot must keep pace with the ball in the perpendicular direction while trying to close on it in the $\mathbf{a} - \mathbf{0}$ direction.

[1] Thus, naturally, I'd forgotten how to do it completely and had to look it up!

If it's speed is too low, it can't keep pace in the perpendicular direction and hence can't intercept the ball.)

Provided the condition for S we derived is satisfied, there are solutions to the above equation for k. This doesn't mean, however, that they're necessarily valid solutions for our problem: we require that the interception happens at $t > 0$, so we need to place a further restriction on k, namely that it's strictly positive (since if $k \leq 0$ then $t = \frac{1}{k}$ is either < 0 or undefined). So:

$$\frac{-\mathbf{u} \bullet \mathbf{a}}{|\mathbf{a}|} \pm \sqrt{S^2 - \left(|\mathbf{u}|^2 - \left(\frac{\mathbf{u} \bullet \mathbf{a}}{|\mathbf{a}|}\right)^2\right)} > 0$$

In other words one of the following happens:

- If $\dfrac{\mathbf{u} \bullet \mathbf{a}}{|\mathbf{a}|} \geq 0$ then:

  $$(+)\sqrt{S^2 - \left(|\mathbf{u}|^2 - \left(\frac{\mathbf{u} \bullet \mathbf{a}}{|\mathbf{a}|}\right)^2\right)} > \frac{\mathbf{u} \bullet \mathbf{a}}{|\mathbf{a}|}$$

  Whence:

  $$S^2 - |\mathbf{u}|^2 > 0 \Leftrightarrow S > |\mathbf{u}|$$

  (This makes sense since if the ball's moving away from the robot then the robot must move faster than it to catch it up.)

- If $\dfrac{\mathbf{u} \bullet \mathbf{a}}{|\mathbf{a}|} < 0$ then the ball's moving towards the robot and there are two possible ways to intercept: move towards it or move away from it. If the robot moves towards the ball (corresponding to taking the positive square root in the equation), then there are no conditions to satisfy (this is the way to minimise the time required to intercept in this case). If the robot moves away from the ball (corresponding to taking the negative square root in the equation), then we have to satisfy

  $$\left|\sqrt{S^2 - \left(|\mathbf{u}|^2 - \left(\frac{\mathbf{u} \bullet \mathbf{a}}{|\mathbf{a}|}\right)^2\right)}\right| < \frac{-\mathbf{u} \bullet \mathbf{a}}{|\mathbf{a}|}$$

  Whence:

  $$S^2 - |\mathbf{u}|^2 < 0 \Leftrightarrow S < |\mathbf{u}|$$

  (This again makes sense since if the ball's moving towards the robot then it will eventually reach it provided the robot isn't moving away from it at a greater speed.)

Finally, we give our equation for t, the minimum time required to intercept (if possible):

$$t = \frac{|\mathbf{a}|^2}{-\mathbf{a} \bullet \mathbf{u} + \sqrt{(\mathbf{a} \bullet \mathbf{u})^2 + |\mathbf{a}|^2(S^2 - |\mathbf{u}|^2)}}$$

Note that we're taking the positive square root here: that's because (as just noted) it's the best option in each case (in the first case, it's the only option). Note also that I inverted the slightly earlier equation for k instead of the final one: that's because it's probably easier to calculate using that, even though it's easier to see what's going on in the rearranged form.

(b) Additional factors which might come into play include things like:

- Which way the robot is facing. It will probably take it some time to turn.
- How quickly the robot can actually accelerate: it might not take very long if we're lucky, but it won't be instantaneous.
- Whether there are any other players in the way which the robot needs to avoid.
- Whether the ball is going to go out of play if it continues on its present course (no point running after it if we can't catch it before it reaches the touchline).
- Whether the speed of the ball itself is actually constant (in practice it will slow down due to friction with the playing surface).
- Whether or not the ball is travelling in a straight line (it might be curving: think *RoboBeckham*™ or something!)

2. (a) The obvious way to do this is with a variant of depth-first search where we mark visited squares. To make it more efficient (we know where the goal is, after all, and we know something about the type of world we're searching), we could use an heuristic approach: when choosing which unexplored neighbouring square to visit at each stage of the recursion, pick the one which is closest to the line between our current position and the goal. Avoiding walls is automatic, since we make sure we only ever visit non-wall squares.

   (b) See separate sheet.

   (c) See separate sheet.

3. See separate sheet.