Advanced Data Structures and Algorithms
Exercise Sheet 4

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1. (a) How many trees are there in a binomial heap that contains 35,515 nodes?
   (b) Illustrate the binomial heap obtained by inserting a node with key 21 and then a node with key 4 in the binomial heap below.
   (c) Illustrate the binomial heap obtained by deleting the node with key 6 from the binomial heap below.

   Answer

   (a) The easiest way to work this out is to convert 35,515 to binary and count the number of 1's, since an n-node binomial heap contains binomial tree B_i iff $b_i = 1$ where $b_k...b_1b_0$ is the binary representation of n. Accordingly, $35515_{10} = 1000101010111011_2$, whence there are 9 trees in the heap in question.
   (b) See separate sheet.
   (c) Likewise.

2. For question, see problem sheet.

   Answer

   See separate sheet.

3. For question, see problem sheet.

   Answer

   (a) Suppose H is a 2-universal family of hash functions as described. We're required to show (by definition) that for each pair of distinct keys $k, \ell \in U$:

   $$\#\{h \in H : h(k) = h(\ell)\} \leq |H|/m$$

   i.e. if h is drawn from H at random then the probability that k and ℓ hash to the same slot is at most $1/m$. Well, as noted in the question, since H is 2-universal, if h is drawn from H at random then the pair $(h(k), h(\ell))$ is equally likely to be any of the m^2 elements of $\{0, ..., m-1\} \times \{0, ..., m-1\}$. So the probability that $h(k) = h(\ell)$ is given by $\frac{m}{m^2}$, since there are m ways for the two to be equal out of m^2 possibilities. (It's obvious why, but for completeness consider the following. To get $h(k) = h(\ell)$, there are $\#\{0, ..., m-1\} = m$ ways to pick $h(k)$ and then $h(\ell)$ must be the same.) Well $\frac{m}{m^2} = \frac{1}{m}$, which is what we needed the probability to be less than or equal to. So we're done.

   (b) Depressingly, the only contribution I've managed to make towards answering this bit is to pedantically observe that $\langle a_0, ..., a_n \rangle$ isn't an n-tuple: it's an $n+1$-tuple. Judging by the rest of the question, it meant to say $\langle a_0, ..., a_{n-1} \rangle$. Similarly for the x vector.

   When it comes to actually answering the question, I could do with some help!

   (c) The same goes for this bit.

4. For question, see problem sheet.

Answer

(a) Since n_i is the number of keys which hash to slot i, the number of collisions in slot i is given by $\binom{n_i}{2}$. So the number of collisions in total is:

$$\sum_{i=0}^{m-1} \binom{n_i}{2}$$

We actually need twice this number, though, because the pairs are *ordered*. A pair of colliding keys k and ℓ get counted twice, once as (k, ℓ) and once as (ℓ, k). So our expression is:

$$2 \sum_{i=0}^{m-1} \binom{n_i}{2}$$

In practice, of course, this is just $\sum_{i=0}^{m-1} ({}^{n_i}P_2)$.

(b) We calculate as follows:

$$\begin{aligned}
& 2 \sum_{i=0}^{m-1} \binom{n_i}{2} \\
= \;& 2 \sum_{i=0}^{m-1} \frac{n_i!}{2(n_i - 2)!} \\
= \;& \sum_{i=0}^{m-1} n_i(n_i - 1) \\
= \;& \left(\sum_{i=0}^{m-1} n_i^2 \right) - \left(\sum_{i=0}^{m-1} n_i \right) \\
= \;& \left(\sum_{i=0}^{m-1} n_i^2 \right) - n && \{\text{since each key in } U \text{ must hash to one of the slots}\} \\
\geq \;& \frac{1}{m} \left(\sum_{i=0}^{m-1} n_i \right)^2 - n && \{\text{using the hint}\} \\
= \;& \frac{n^2}{m} - n && \{\text{since each key in } U \text{ must hash to one of the slots}\} \\
= \;& n^2 \left(\frac{1}{m} - \frac{1}{n} \right)
\end{aligned}$$

Proving the hint

TODO

(c) There are $\#U \times \#U = n^2$ pairs of keys (k, ℓ), of which we just proved at least $n^2 \left(\frac{1}{m} - \frac{1}{n} \right)$ are such that k and ℓ are distinct and $h(k) = h(\ell)$. Thus the probability that k and ℓ are distinct and $h(k) = h(\ell)$ for any particular pair of keys satisfies:

$$Pr[h(k) = h(\ell)] \geq \frac{n^2 \left(\frac{1}{m} - \frac{1}{n} \right)}{n^2} = \frac{1}{m} - \frac{1}{n}$$

(d) TODO